MAT320/640, Fall 2021 **Renato Ghini Bettiol**

Homework Set 1

DUE: SEP 13, 2021 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. [Ross, Exercise 1.5] Prove (by induction) that, for any $n \in \mathbb{N}$,
$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$
2. Prove that $\sqrt{23}$ is not a rational number.
3. Is $\sqrt{18 + 2\sqrt{17}} - \sqrt{17}$ a rational number or not? Justify.
Hint: Don't jump to any conclusions too fast here.
Note that $\left(\sqrt{18 + 2\sqrt{17}}\right)^2 = 18 + 2\sqrt{17} = 17 + 2\sqrt{17} + 1 = \ldots$

Solutions

Text in blue represents side comments that are not integral parts of proofs, but address issues that some students might have had difficulties with in their attempted solutions.

1. *Proof.* For each $n \in \mathbb{N}$, let P_n be the following statement:
$$(P_n) \qquad 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.$$
Basis. Statement (P_1) is $1 + \frac{1}{2^1} = 2 - \frac{1}{2^1}$, which is trivially true, since both sides evaluate to $\frac{3}{2}$.
Step. Assuming that statement (P_n) holds, we now prove that statement (P_{n+1}) holds:
$$\begin{aligned}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right) + \frac{1}{2^{n+1}} &\overset{(P_n)}{=} \left(2 - \frac{1}{2^n}\right) + \frac{1}{2^{n+1}} \\ &= 2 - \frac{2}{2^{n+1}} + \frac{1}{2^{n+1}} \\ &= 2 - \frac{1}{2^{n+1}}.\end{aligned}$$
□

2. *Proof.* Consider the polynomial $p(x) = x^2 - 23$, and note that $\sqrt{23}$ is, by definition, the only positive root of $p(x)$. By the Rational Zeros Theorem (Lecture 2), since $p(x)$ is monic and has constant coefficient $a_0 = 23$, any rational root of $p(x)$ must be among the divisors of 23. As 23 is a prime number, its divisors are ± 1 and ± 23. By direct inspection, $p(\pm 1) \neq 0$ and $p(\pm 23) \neq 0$, therefore $\sqrt{23} \notin \mathbb{Q}$ is not rational. $\square$

3. Following the hint, and "completing the square", we find:

$$\left(\sqrt{18 + 2\sqrt{17}}\right)^2 = 18 + 2\sqrt{17} = 17 + 2\sqrt{17} + 1 = \left(\sqrt{17} + 1\right)^2,$$

therefore $\sqrt{18 + 2\sqrt{17}} = \sqrt{17} + 1$, since both are positive numbers with the same square, and hence the following number is clearly rational (actually, an integer!):

$$\sqrt{18 + 2\sqrt{17}} - \sqrt{17} = 1 \in \mathbb{Q}.$$