(Review for the Final Exam)

#1 Consider the function $f(x) = C(5-x^2)$, $-\sqrt{5} \le x \le \sqrt{5}$.

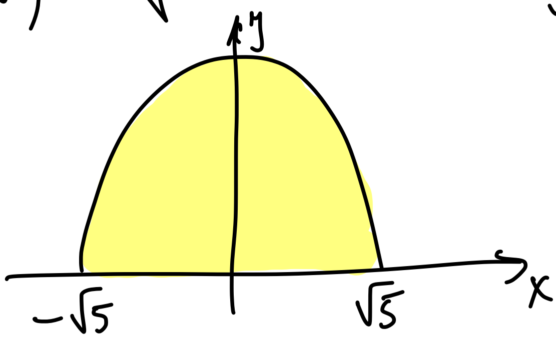
a) Find $C > 0$ such that $f(x)$ is a p.d.f.

b) Compute $E(X)$ if X has p.d.f $f(x)$.

c) Compute $Var(X)$ if X has p.d.f. $f(x)$.

d) Find the p.d.f of $Y = X^2$.

a) $f(x) \ge 0$, $1 = \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\sqrt{5}}^{\sqrt{5}} C(5-x^2)\,dx =$



$$= C\left(5x - \frac{x^3}{3}\right)\Big|_{-\sqrt{5}}^{\sqrt{5}} =$$

$$= C\left(5\sqrt{5} - \frac{5\sqrt{5}}{3} - \left(-5\sqrt{5} + \frac{5\sqrt{5}}{3}\right)\right)$$

$$= C\left(10\sqrt{5} - \frac{10\sqrt{5}}{3}\right) = \frac{20\sqrt{5}}{3} C$$

$$\Rightarrow C = \frac{3}{20\sqrt{5}} = \boxed{\frac{3\sqrt{5}}{100}}$$

b) $E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-\sqrt{5}}^{\sqrt{5}} x \frac{3\sqrt{5}}{100}(5-x^2)\,dx =$

$$= \frac{3\sqrt5}{100}\int_{-\sqrt5}^{\sqrt5} 5x - x^3\,dx = \boxed{0}$$

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x)\,f(x)\,dx$$

c) $$Var(X) = E(X^2) - \underbrace{E(X)}_{=0}^2 = E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

$$= \frac{3\sqrt5}{100}\int_{-\sqrt5}^{\sqrt5} x^2(5-x^2)\,dx = \frac{3\sqrt5}{50}\int_0^{\sqrt5} 5x^2 - x^4\,dx$$

$$= \frac{3\sqrt5}{50}\left(\frac{5x^3}{3} - \frac{x^5}{5}\right)\Big|_0^{\sqrt5} = \frac{3\sqrt5}{50}\left(\frac{25\sqrt5}{3} - \frac{\cancel{25}^{5}\sqrt5}{\cancel{5}}\right)$$

$$= \frac{3\sqrt5}{50}\,\frac{25\sqrt5 - 15\sqrt5}{3} = \frac{\cancel{3}\sqrt5}{50}\cdot\frac{10\sqrt5}{\cancel{3}} = \boxed{1}$$

d) $$Y = X^2 \qquad f_Y(y) = ?$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt y \le X \le \sqrt y)$$

$$= \int_{-\sqrt y}^{\sqrt y} \frac{3\sqrt5}{100}(5-x^2)\,dx = \frac{3\sqrt5}{100}\cdot 2\int_0^{\sqrt y} 5 - x^2\,dx$$

$$= \frac{3\sqrt5}{50}\left(5x - \frac{x^3}{3}\right)\Big|_0^{\sqrt y} = \frac{3\sqrt5}{50}\left(5\sqrt y - \frac{y\sqrt y}{3}\right) =$$

$$= \frac{3\sqrt{5}}{50}\,\frac{15\sqrt{y} - y\sqrt{y}}{3} = \frac{\sqrt{5}}{50}\left(15y^{1/2} - y^{3/2}\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{\sqrt{5}}{50}\left(\frac{15}{2}y^{-1/2} - \frac{3}{2}y^{1/2}\right)$$

$$\boxed{= \frac{3\sqrt{5}}{100}\left(\frac{5}{\sqrt{y}} - \sqrt{y}\right).}$$

Note: Check that indeed this is a p.d.f., that is,

$$\int_0^5 f_Y(y)\,dy = 1$$

#2 A fair coin is tossed 100 times.

a) What is the expected number of heads?

b) What is the standard deviation in the number of heads?

c) Write an exact formula for the probability that the observed number of heads deviates from the expected number by 3 or more standard deviations.

d) Use Chebyshev's inequality to bound this probability.

X = # of heads observed when tossing 100 times a fair coin

$$X \sim \text{Binomial}\left(100, \tfrac{1}{2}\right) \qquad n = 100, \quad p = \tfrac{1}{2}.$$

a) $$E(X) = n \cdot p = 100 \cdot \tfrac{1}{2} = \boxed{50}$$

b) $$\sigma(X) = \sqrt{Var(X)} = \sqrt{n\,p(1-p)} = \sqrt{100 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}} = \sqrt{25} = \boxed{5}$$

c) $$P(|X-50| \geq 15) = 1 - P(35 \leq X \leq 65)$$

$$= 1 - \sum_{i=35}^{65} \binom{100}{i} \left(\frac{1}{2}\right)^{i} \left(\frac{1}{2}\right)^{100-i} = 1 - \sum_{i=35}^{65} \binom{100}{i} \frac{1}{2^{100}}$$

$$= \boxed{1 - \frac{1}{2^{100}} \sum_{i=35}^{65} \binom{100}{i}} \cong 0.001789\ldots$$

d) $$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

$$P(|X-50| \geq 15) \leq \frac{5^2}{15^2} = \frac{\cancel{5^2}}{3^2 \cdot \cancel{5^2}} = \boxed{\frac{1}{9}} \cong 0.111\ldots$$

#3. Suppose that a certain car component, on average, fails after 100,000 miles; and is exponentially distributed.

a) What is the probability that it fails before 50,000 miles?

b) If a car owner has driven over 50,000 miles without the component failing, what is the probability that it fails before reaching 100,000 mi?

c) What is the standard deviation of the number of miles until the component fails?

X = thousands of miles until the component fails.

$X \sim \text{Exponential}\left(\frac{1}{100}\right)$, $\lambda = \frac{1}{100}$ because $E(X) = 100$

a) $$P(X \leq 50) = \int_0^{50} \frac{1}{100} e^{-\frac{1}{100}x}\, dx = \frac{1}{\cancel{100}} \left(\frac{e^{-\frac{1}{100}x}}{-\frac{1}{\cancel{100}}} \right)\Bigg|_0^{50}$$

$$= -\left(e^{-\frac{1}{2}} - e^{0}\right) = 1 - e^{-1/2} = \boxed{1 - \frac{1}{\sqrt{e}}} \cong 0.3935$$

b) $$P(X \leq 100 \mid X \geq 50) = 1 - P(X \geq 100 \mid X \geq 50)$$

$$= 1 - P(X \geq 50) = P(X \leq 50) = \boxed{1 - \frac{1}{\sqrt{e}}} \cong 0.3935$$

Memorylessness:

$$P(X \geq a + b \mid X \geq a) = P(X \geq b)$$

c) $\sigma(X) = \sqrt{Var(X)} = \sqrt{1/\lambda^2} = \frac{1}{\lambda} = \boxed{100}$

$X \sim Exponential\left(\frac{1}{100}\right)$

100, 000 miles.

#4 Suppose X is a cont. random variable with moment generating function $M(t) = \frac{1}{(1-t)^2}$.

a) Find the expected value of X

b) Find the standard deviation of X.

$M(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x)\,dx$

$M'(0) = E(X)$ $\quad M''(0) = E(X^2)$

a) $E(X) = M'(0) = \frac{d}{dt}\left(\frac{1}{(1-t)^2}\right)\Big|_{t=0}$

$= -2(1-t)^{-3}\cdot(-1)\Big|_{t=0} = \boxed{2}$

b) $\sigma(X) = \sqrt{Var(X)}$ $\qquad Var(X) = E(X^2) - E(X)^2$

$E(X^2) = M''(0) = \frac{d}{dt}\left(2(1-t)^{-3}\right)\Big|_{t=0} =$

$= -6(1-t)^{-4}\cdot(-1)\Big|_{t=0} = \boxed{6}$

$$Var(X) = E(X^2) - E(X)^2 = 6 - (2)^2 = \boxed{2}$$

$$\sigma(X) = \sqrt{Var(X)} = \boxed{\sqrt{2}}$$

#5 Consider the following jointly distributed discrete random variables:

$X \backslash Y$	0	1	Marginal for X
0	1/4	1/4	1/2
1	1/4	1/4	1/2
Marginal for Y	1/2	1/2	

$Z \backslash W$	0	1	Marginal for Z
0	0	1/2	1/2
1	1/2	0	1/2
Marginal for W	1/2	1/2	

a) Are X and Y independent? Compute $Cov(X,Y)$.

b) Are Z and W independent? Compute $Cov(Z,W)$.

a) X and Y are independent if and only if

$$P(X=x, Y=y) = P(X=x)\cdot P(Y=y) \quad \forall x,y \in \{0,1\}$$

Since the above holds, it follows that X, Y are independent. Thus, $\underline{Cov(X,Y) = 0}$.

b) $P(Z=0, W=0) = 0 \qquad P(Z=0) = \frac{1}{2} = P(W=0)$

$0 = P(Z=0, W=0) \neq P(Z=0) \cdot P(W=0) = 1/4$

Therefore Z and W are <u>not</u> independent.

$Cov(Z,W) = E\big((Z-E(Z))(W-E(W))\big) = E(ZW) - E(Z)E(W)$

$$\left.\begin{aligned} E(Z) &= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2} \\ E(W) &= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2} \\ E(ZW) &= 0 \cdot 1 + 1 \cdot 0 = 0 \end{aligned}\right\} = 0 - \frac{1}{2} \cdot \frac{1}{2} = \boxed{-\frac{1}{4}}$$