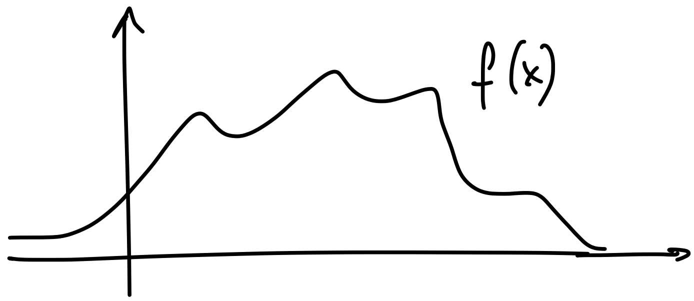
## Simulating a random variable with prescribed pdf:




Q: How to create a rand. var $X$ with p.d.f. given by $f(x)$, using as input $U \sim \text{Uniform}((0,1))$.

Note: This is very important in computer science & data science.

**Prop:** Given $f(x)$, let $F(x) = \int_{-\infty}^{x} f(t)\,dt$. Define $X = F^{-1}(U)$, where $U \sim \text{Uniform}((0,1))$. Then $X$ is a random variable with pdf $f(x)$.

**Pf:**

$$F_X(x) = P(X \le x) = P(F^{-1}(U) \le x) \overset{F \text{ is monot. increasing}}{=}$$

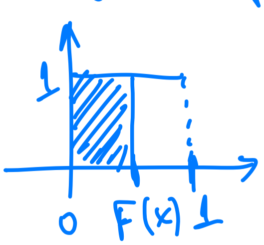
$$= P\big(F(F^{-1}(U)) \le F(x)\big) = P(U \le F(x))$$

$U \sim \text{Unif}((0,1))$

$$\overset{}{=} F(x).$$

Differentiating both sides in $x$:

$$f_X(x) = f(x), \text{ as desired.} \qquad \square$$



**Example:** Simulate an exponential random variable with $\lambda = 1$, using as input $U \sim \mathrm{Unif}([0,1])$.

$$f(x) = e^{-x} \quad \leftarrow \text{prescribed p.d.f.}$$

$$F(x) = \int_{-\infty}^{x} e^{-t}\,dt = \dots = 1 - e^{-x} \quad \leftarrow \text{prescribed c.f.d.}$$

$$y = F(x) = 1 - e^{-x} \Rightarrow e^{-x} = 1 - y \Rightarrow x = -\log(1-y)$$

$$F^{-1}(y) = -\log(1-y) = \log\frac{1}{1-y}.$$

$$X = F^{-1}(U) = \log\left(\frac{1}{1-U}\right) \text{ has the desired p.d.f. } f_X(x) = e^{-x}$$

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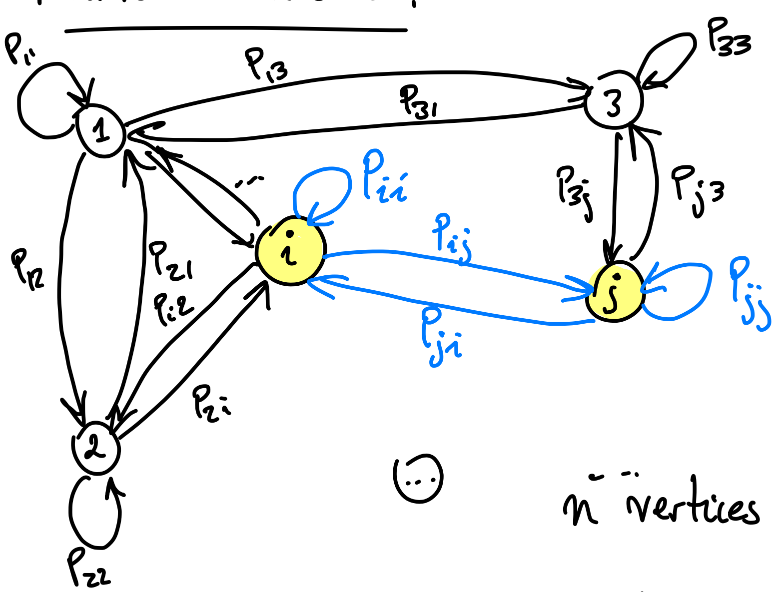
**Markov Chains:**




"Transition Diagram"

$$P_{ij} = P\big(\underset{\text{next state}}{X_{k+1} = j} \,\big|\, \underset{\text{current state}}{X_k = i}\big)$$

$$\underset{\text{Markov hypothesis}}{=} P\big(X_{k+1} = j \mid X_k = i, \underbrace{X_{k-1} = i_{k-1}, \dots, X_0 = i_0}_{\text{"past"}}\big),\ \forall k.$$

"Past is irrelevant, only current state suffices to know prob. distr. of where system will be in next step!"

Convenient to display $P_{ij}$'s in "Transition matrix"

$$
\begin{array}{l}
\text{current state } 1 \rightarrow \\
\text{current state } 2 \rightarrow \\
\\
\text{current state } n \rightarrow
\end{array}
\quad
P = \begin{pmatrix}
P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\
P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \\
\vdots & \vdots & \vdots & & \\
P_{n1} & P_{n2} & P_{n3} & \cdots & P_{nn}
\end{pmatrix}
$$

Adding the numbers in each of these lines gives 1.

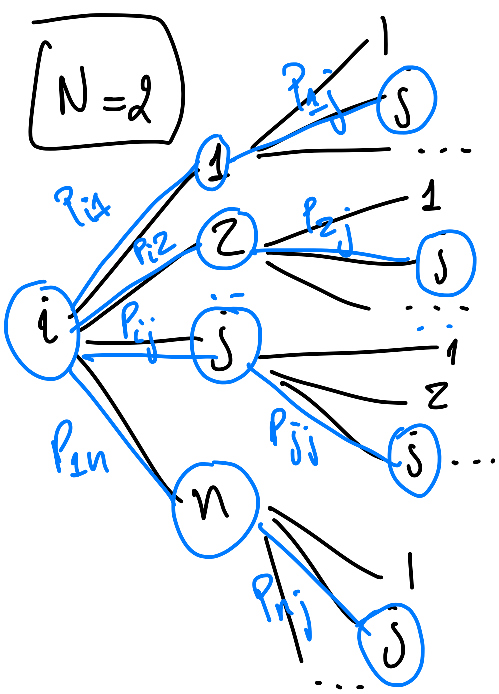
"Stochastic matrix"

<u>More rigorously</u>: A <u>Markov chain</u> is a sequence of random variables $\{X_k\}$ taking values on $\{1, 2, \dots, n\}$ that satisfies the "Markov hypothesis" written above.

$X_k = i$ means that at time $k$ the system is in state $i$

<u>Basic questions</u>:

<u>Q1</u>: What is the probability of going from state $i$ to state $j$ after $N$ steps?




$$P_{ij}^{(2)} = P_{i1}P_{1j} + P_{i2}P_{2j} + \dots + P_{1n}P_{nj}$$

$$= \sum_{l=1}^{n} P_{il}P_{lj}$$

$\leftarrow$ This is exactly the $(i,j)$ entry of the matrix $P^2$

## Chapman–Kolmogorov Equation:

$$P_{ij}^{(N)} = \sum_{l=1}^{n} P_{il}^{(r)} P_{lj}^{(N-r)} \quad \text{for any } 0<r<N$$

transition prob. from state $i$ to state $j$ in $N$ steps

(e.g. setting $r=N-1$:

$$P_{ij}^{(N)} = \sum_{l=1}^{n} P_{il}^{(N-1)} P_{lj}$$
)

$(i,j)$ entry in the matrix $P^{(N)}$.

<u>Q2</u>: With what frequency is the system in a given state? (As number of steps goes to $+\infty$, in which state is the system most/least likely to be?)

$$(\star)_j \quad \pi_j = \lim_{N\to\infty} P_{ij}^{(N)}$$

<u>Note</u>: "Initial state" $i$ does not matter here.

From Linear Algebra, we know how to compute powers $P^{(N)}$ of a matrix $P$ for all $N$ if we know the eigenvalues and eigenvectors of $P$.

<u>Thm</u>. For an <u>ergodic</u> Markov chain, the limits $(\star)_j$ exist for all $1\le j\le n$, and the resulting $\pi_j$ are the unique non-negative solutions of $\pi_j = \sum_{l=1}^{n} \pi_l P_{lj}$, $\sum_{j=1}^{n} \pi_j = 1$

i.e. st $P_{ij}^{(N)} > 0$ for all $i,j$ for some $N>0$.