Moment generating functions:

Def: The moment generating function of a random variable X is the function $M: I \to \mathbb{R}$ given by

$$M(t) = E\left(e^{tX}\right)$$

Note: If X is a discrete random variable, then

$$M(t) = \sum_{x} e^{tx} \cdot p(x)$$

If X is a cont. random variable, then

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)\,dx$$

Example: $X \sim \text{Binomial}(n,p) \Rightarrow p(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$M(t) = E\left(e^{tX}\right) = \sum_{x=0}^{n} e^{tx} p(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} \left(e^t p\right)^x (1-p)^{n-x} = \left(pe^t + 1 - p\right)^n$$

Moments of a random variable:

1st moment of X : $E(X) = M'(0)$

2nd —— " —— : $E(X^2) = M''(0)$

3rd —— " —— : $E(X^3) = M'''(0)$

nth —— " —— : $E(X^n) = M^{(n)}(0)$

Moment generating function: $M(t) = E(e^{tX})$

$$M(0) = E(e^{0X}) = E(1) = 1.$$

$$M'(t) = \frac{d}{dt} E(e^{tX}) = E\left(\frac{d}{dt} e^{tX}\right) = E\left(Xe^{tX}\right)$$

Evaluate the above at $t=0$:

$M'(0) = E(X)$. ← 1st moment of X

$$M''(t) = \frac{d}{dt} M'(t) = \frac{d}{dt} E(Xe^{tX}) = E\left(\frac{d}{dt} Xe^{tX}\right)$$

$$= E\left(X^2 \cdot e^{tX}\right)$$

Evaluate the above at $t=0$;

$$M''(0) = E(X^2).$$ ← 2^{nd} moment of X

By induction on n:

$$M^{(n)}(t) = E(X^n \cdot e^{tX})$$

Evaluating at $t=0$:

$$M^{(n)}(0) = E(X^n)$$ ← n^{th} moment of X.

$\boxed{\text{\underline{Upshot}: } n^{th} \text{ moment of } X \text{ is } E(X^n) = M^{(n)}(0).}$

$\boxed{\begin{array}{l}\underline{\text{Recall}}: E(X) = np \\ \quad Var(X) = np(1-p)\end{array}}$

Back to the Binomial example:

$$X \sim \text{Binomial}(n,p) \Rightarrow M(t) = (pe^t + 1 - p)^n$$

$$M'(t) = n(pe^t + 1 - p)^{n-1} \cdot (pe^t)$$

matches what we knew from before!

$$M'(0) = n(\cancel{p} + 1 - \cancel{p})^{n-1} \cdot p = n \cdot p = E(X).$$

$$M''(t) = n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 +$$
$$+ n(pe^t + 1 - p)^{n-1} \cdot (pe^t)$$

$$M''(0) = n(n-1)p^2 + np = np\left((n-1)p+1\right) = E(X^2)$$

$$Var(X) = E(X^2) - E(X)^2 = n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = np - np^2 = \underline{np(1-p)}$$

Matches what we Knew from before!

Ex: Exponential Random Variable

$X \sim \text{Exponential}(\lambda)$ $\qquad f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$M(t) = E\left(e^{tX}\right) = \int_0^{+\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{+\infty} e^{-(\lambda - t)x}\, dx = \lambda \cdot \lim_{b\to\infty} \int_0^b e^{-(\lambda-t)x}\, dx$$

$$= \lambda \lim_{b\to\infty} \left. \frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right|_0^b$$

$$= \lambda \lim_{b\to\infty} \underbrace{\frac{e^{-(\lambda-t)b}}{-(\lambda-t)}}_{\to\, 0 \text{ if } \lambda - t > 0} - \frac{1}{-(\lambda-t)} = \frac{\lambda}{\lambda - t},$$

$$\underline{\text{if } t < \lambda}$$

$$M(t) = \frac{\lambda}{\lambda - t} \quad \text{if} \quad t < \lambda.$$

Recall: $E(X) = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$

$$M'(t) = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right) = \frac{\lambda}{(\lambda - t)^2} \Rightarrow M'(0) = \frac{1}{\lambda} = E(X) \checkmark$$

$$M''(t) = \frac{d}{dt} \frac{\lambda}{(\lambda - t)^2} = \frac{2\lambda}{(\lambda - t)^3} \Rightarrow M''(0) = \frac{2}{\lambda^2} = E(X^2) \checkmark$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} \checkmark$$

Theorem: Suppose that $M_X(t) = E(e^{tX})$ is well-def. in a neighborhood of $t = 0$. Then the distr. of X is uniquely determined by the function $M_X(t)$.

(Really by its Taylor Series at $t=0$)

Ex: $M_X(t) = \frac{1}{1-t}$ if $t < 1$.

$$\Rightarrow X \sim \text{Exponential}(1).$$

Because X has the same moment generating function as an exponential random variable with $\lambda = 1$.

Moment generating function of normal random variables:

Recall: $\int_{-\infty}^{+\infty} e^{-s^2/2}\, ds = \sqrt{2\pi}$. (See video linked below @ 3:30)

$$X \sim \text{Normal}(\mu, \sigma^2) \implies f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Say $Z \sim \text{Normal}(0,1)$, then $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$$M_Z(t) = E\left(e^{tZ}\right) = \int_{-\infty}^{+\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2} + tz}\, dz$$

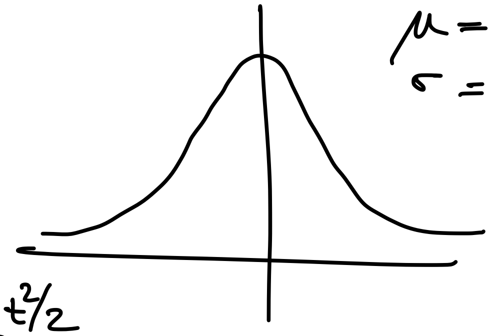
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(z-t)^2}{2} + \frac{t^2}{2}}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{(z-t)^2}{2}} \cdot \underbrace{e^{\frac{t^2}{2}}}\, dz = \frac{e^{t^2/2}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{(z-t)^2}{2}}\, dz}_{=\sqrt{2\pi}}$$

$$\begin{cases} s = z - t \\ ds = dz \end{cases}$$

$$= e^{t^2/2}.$$

$$\boxed{\text{Upshot: } M_Z(t) = e^{t^2/2}}$$



$\mu = 0$ ✓
$\sigma = 1$ ✓

$$M_Z'(t) = e^{t^2/2} \cdot t$$

$$M_Z''(t) = e^{t^2/2} \cdot t^2 + e^{t^2/2} = (t^2+1)e^{t^2/2}$$

$$E(Z) = M_Z'(0) = 0 \checkmark$$

$$E(Z^2) = M_Z''(0) = 1 \checkmark$$

$$\operatorname{Var}(Z) = E(Z^2) - E(Z)^2$$
$$= 1 - 0 = 1 \checkmark$$

Q: How about $X \sim \text{Normal}(\mu, \sigma^2)$?

$$X = \mu + \sigma Z$$

$$M_X(t) = E\left(e^{tX}\right) = E\left(e^{t(\mu+\sigma Z)}\right) = E\left(e^{t\mu} \cdot e^{t\sigma Z}\right)$$

$$= e^{t\mu} \underbrace{E\left(e^{t\sigma Z}\right)}_{M_Z(t\sigma)} = e^{t\mu} M_Z(t\sigma)$$

$$= e^{t\mu} \cdot e^{(t\sigma)^2/2} = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

Check: $M_X'(0) = \mu$, $M_X''(0) = \mu^2 + \sigma^2$

Moment generating function of a sum:

If X, Y are independent random variables, then

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right) = E\left(e^{tX} \cdot e^{tY}\right)$$

$$\overset{\text{independence}}{=} E\left(e^{tX}\right) \cdot E\left(e^{tY}\right) = M_X(t) \cdot M_Y(t).$$

E.g., $X_1 \sim \text{Normal}(\mu_1, \sigma_1^2)$, $X_2 \sim \text{Normal}(\mu_2, \sigma_2^2)$ indep.

Q: How is $X_1 + X_2$ distributed?

A1: Using what we did before, $f_{X_1+X_2} = f_{X_1} \underset{\text{convolution}}{*} f_{X_2}$

A2: Using Moment generating functions:

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \cdot e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$$

$$= e^{(\mu_1+\mu_2)t + \frac{t^2}{2}(\sigma_1^2 + \sigma_2^2)}$$

$$= M_S(t), \text{ where } S \sim \text{Normal}\left(\mu_1+\mu_2, \sigma_1^2 + \sigma_2^2\right)$$

Therefore, $X_1 + X_2$ is distributed in the same way as S (by the Theorem above).

$$X_1 + X_2 \sim \text{Normal}\left(\mu_1 + \mu_2,\ \sigma_1^2 + \sigma_2^2\right)$$