Expected value of functions of random variables:

Recap: if X is a continuous random variable

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

Def: If X, Y are cont. rand. var, then

$$E(g(X,Y)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

(cf. if X, Y are discrete rand. var., then

$$E(g(X,Y)) = \sum_x \sum_y g(x,y)\, p(x,y)$$
)

Prop: $E(\cdot)$ is linear (Note: we proved this already for discrete random variables)

Pf: X, Y cont. random variables, $a, b \in \mathbb{R}$

Want to show: $E(aX + bY) = a E(X) + b E(Y)$.

$$E(aX+bY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (ax+by)\, f_{X,Y}(x,y)\, dx\,dy$$

$g(x,y) = ax+by$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[a x f_{X,Y}(x,y) + b y f_{X,Y}(x,y)\right] dx\,dy$$

$$= a \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{X,Y}(x,y)\, dx\,dy + b \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y f_{X,Y}(x,y)\, dx\,dy$$

$$\left.\begin{aligned} f_X(x) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy \\ f_Y(y) &= \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx \end{aligned}\right\}$$

$$= a \underbrace{\int_{-\infty}^{+\infty} x f_X(x)\,dx}_{=E(X)} + b \underbrace{\int_{-\infty}^{+\infty} y f_Y(y)\,dy}_{=E(Y)}$$

$$= a E(X) + b E(Y). \qquad \square$$

<u>Recall</u>: $\mathrm{Cov}(X,Y) = E\big((X-E(X))(Y-E(Y))\big)$

$$= E(XY) - E(X)E(Y).$$

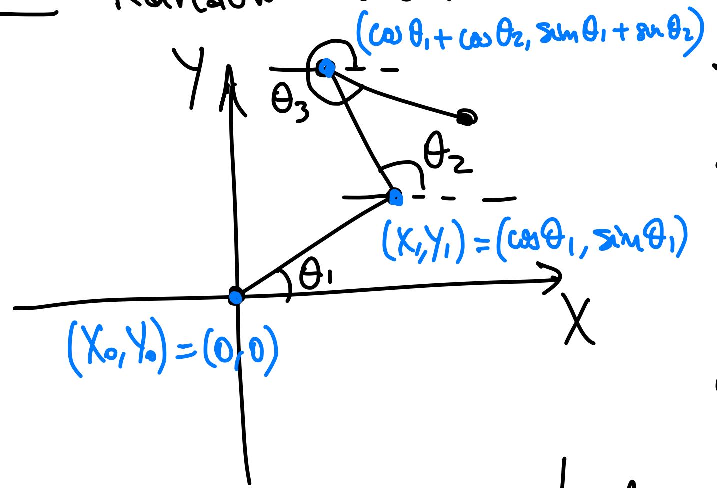
<u>Cor</u>: If X,Y are <u>independent</u>, then

$$E(XY) = E(X)\cdot E(Y).$$

Also, if g,h are continuous functions, then

$$E(g(X)h(Y)) = E(g(X))\,E(h(Y)).$$

Ex: Random walk in the plane, n steps:



Take steps of length 1 in a random and independent direction:

$$\theta_i \sim \text{Unif}([0,2\pi])$$

at each step $1 \le \theta \le n$.

Q: Find the expected value of the square of the distance to the origin after n steps.

Position after n steps: $\left(\sum_{i=1}^{n} \cos\theta_i, \sum_{i=1}^{n} \sin\theta_i \right)$

$$E\left(\left(\sum_{i=1}^{n} \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} \sin\theta_i\right)^2 \right) \overset{\text{linearity}}{=}$$

$$= E\left(\left(\sum_{i=1}^{n} \cos\theta_i\right)^2 \right) + E\left(\left(\sum_{i=1}^{n} \sin\theta_i\right)^2 \right)$$

$$= E\left(\sum_{i=1}^{n} \cos^2\theta_i + \sum_{i\neq j} \cos\theta_i \cos\theta_j \right)$$

$$+ E\left(\sum_{i=1}^{n} \sin^2\theta_i + \sum_{i\neq j} \sin\theta_i \sin\theta_j \right)$$

linearity

$$\stackrel{\text{linearity}}{=} \sum_{i=1}^{n} E(\cos^2\theta_i) + \sum_{i\neq j} E(\cos\theta_i \cos\theta_j)$$

$$+ \sum_{i=1}^{n} E(\sin^2\theta_i) + \sum_{i\neq j} E(\sin\theta_i \sin\theta_j)$$

$$= \sum_{i=1}^{n} \Big(E(\cos^2\theta_i) + E(\sin^2\theta_i)\Big) + \sum_{i\neq j} E(\cos\theta_i \cos\theta_j) + E(\sin\theta_i \sin\theta_j)$$

$$= \sum_{i=1}^{n} E(\underbrace{\cos^2\theta_i + \sin^2\theta_i}_{1}) + \sum_{i\neq j} E(\cos\theta_i)E(\cos\theta_j) + E(\sin\theta_i)E(\sin\theta_j)$$

$$\left\{\begin{array}{l} E(\cos\theta_i \cos\theta_j) = E(\cos\theta_i)E(\cos\theta_j) \\ E(\sin\theta_i \sin\theta_j) = E(\sin\theta_i)E(\sin\theta_j) \end{array}\right. \quad \text{by } \underline{\underline{\text{indep.}}}$$

$$= n + 0$$

$$\left\{\begin{array}{l} E(\cos\theta_i) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta_i \, d\theta_i = 0 \\ E(\sin\theta_i) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta_i \, d\theta_i = 0 \end{array}\right.$$

$$= n.$$

$\underline{A}$: $E(\text{square dist. to origin}) = \#\text{ steps} = n.$

Conditional expectation:

Discrete random variables:

Probability mass function of X given that $Y=y$:

$$p_{X|Y}(x,y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

Continuous random variables:

Probability density function of X given that $Y=y$:

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{if} \quad f_Y(y) > 0$$

Conditional expectation:

Discrete r.v.:

$$E(X \mid Y=y) = \sum_x x \underbrace{P(X=x \mid Y=y)}_{p_{X|Y}(x,y)} = \sum_x x\, p_{X|Y}(x,y)$$

Cont. r.v.:

$$E(X \mid Y=y) = \int_{-\infty}^{+\infty} x\, f_{X|Y}(x,y)\, dx$$

Conditional Expectation as a Random Variable:

$g(y) = E(X|Y=y)$ is a function of y

This defines a random variable $g(Y)$; denoted

$E(X|Y)$. ← is itself a random variable, that takes value $E(X|Y=y)$ when Y takes value y

Q: $E(E(X|Y)) = ?$

A: $\boxed{\text{Conditional Expectation Formula} \quad E(E(X|Y)) = E(X).}$ (holds for both discrete or cont. r.v.)

Pf: Assuming X, Y discrete;

$$E(E(X|Y)) = \sum_y E(X|Y=y)\, p_Y(y)$$

$$= \sum_y \sum_x x\, p_{X|Y}(x,y)\, p_Y(y)$$

$$= \sum_y \sum_x x \, \frac{p_{X,Y}(x,y)}{\cancel{p_Y(y)}} \cdot \cancel{p_Y(y)}$$

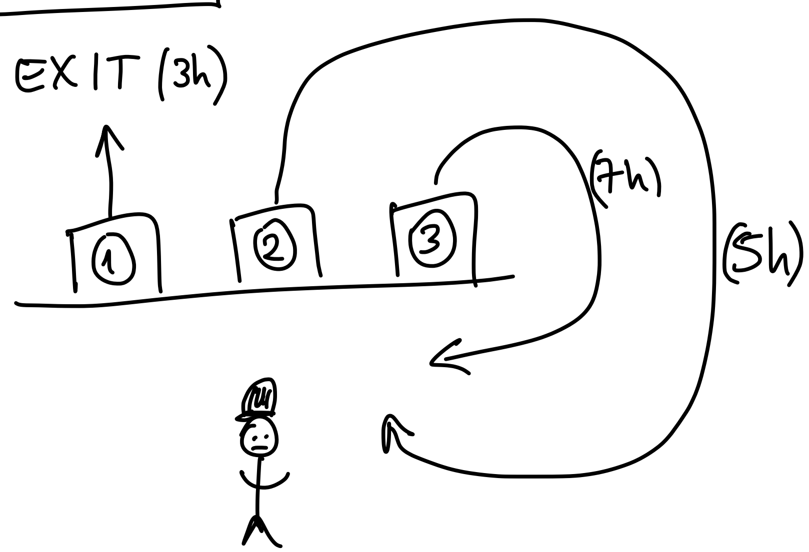
$$= \sum_x x \left(\underbrace{\sum_y p_{X,Y}(x,y)}_{p_X(x)} \right) = \sum_x x \, p_X(x)$$

$$= E(X).$$

$\square$

(Compare with:
$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$)

Exercise: A miner is trapped in a mine with 3 doors.



Q: What is the expected number of hours that miner will take to reach exit?

X = # hours until exit

Y = door chosen.

Want:
$$E(X) = E(X|Y=1)P(Y=1) + E(X|Y=2)P(Y=2) + E(X|Y=3)P(Y=3)$$

$$= \frac{1}{3}\Big(E(X \mid Y=1) + E(X \mid Y=2) + E(X \mid Y=3)\Big)$$

$$\left.\begin{aligned} E(X \mid Y=1) &= 3 \\ E(X \mid Y=2) &= 5 + E(X) \\ E(X \mid Y=3) &= 7 + E(X) \end{aligned}\right\}$$

$$= \frac{1}{3}\Big(3 + 5 + E(X) + 7 + E(X)\Big)$$

$$= \frac{1}{3}\Big(15 + 2E(X)\Big)$$

Thus: $E(X) = 5 + \frac{2}{3}E(X)$.

$$\frac{1}{3}E(X) = 5 \implies \boxed{E(X) = 15}$$