Jointly distributed Random Variables

X, Y random variables

If X, Y discrete:

$p(x,y) = P(X=x, Y=y)$ "joint prob. mass function"

$X \backslash Y$	y_1	y_2	$\cdots$	y_m	marginal prob. on X
x_1	$p(x_1,y_1)$	$p(x_1,y_2)$	$\cdots$	$p(x_1,y_m)$	$P(X=x_1)$
x_2	$p(x_2,y_1)$	$p(x_2,y_2)$	$\cdots$	$p(x_2,y_m)$	$P(X=x_2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_n	$p(x_n,y_1)$	$p(x_n,y_2)$	$\cdots$	$p(x_n,y_m)$	$P(X=x_n)$
marginal prob on Y	$P(Y=y_1)$	$P(Y=y_2)$	$\cdots$	$P(Y=y_m)$	

$X = x_i$ and $Y = y_1$ or y_2 or $\dots$ y_m

Marginals: 1) $P(X=x_i) = \sum_{j=1}^{m} p(x_i, y_j) = p_X(x_i)$

this is the prob. mass function for X.

2) $P(Y=y_j) = \sum_{i=1}^{n} p(x_i, y_j) = p_Y(y_j)$

this is the prob. mass function for Y.

Example: Probabilities that a pedestrian gets hit by a car while crossing at an intersection with a traffic light.

H = Hit / Not hit L = Red / Yellow / Green

H \ L	Red	Yellow	Green	Marginal on H
Not hit	0.198	$\boxed{0.09}$	0.14	0.428
Hit	0.002	0.01	0.56	0.572
Marginal on L	0.2	0.1	0.7	

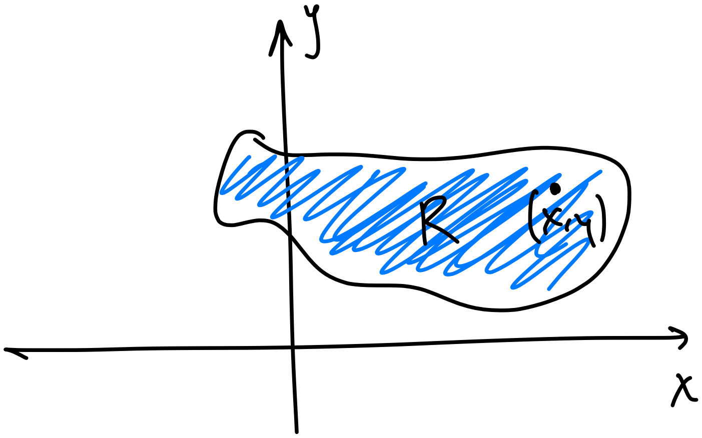
$P(L = \text{Yellow}, H = \text{Not hit}) = 0.09$

How about $P(H = \text{Not hit} \mid L = \text{Yellow}) = ?$

$$P(H = \text{Not hit} \mid L = \text{Yellow}) = \frac{P(H = \text{Not hit}, L = \text{Yellow})}{P(L = \text{Yellow})}$$

$$= \frac{0.09}{0.1} = \underline{0.9}$$

If X, Y are continuous random variables:



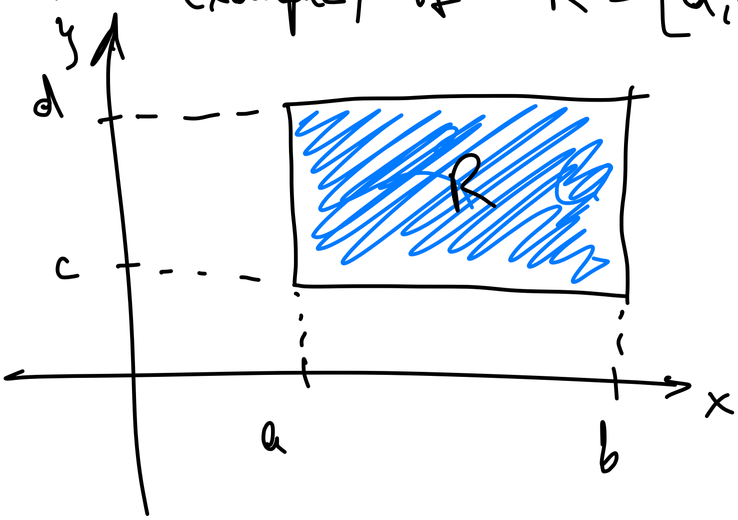
Joint prob. density function:

$f_{X,Y}(x,y) \geq 0$, such that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = 1.$$

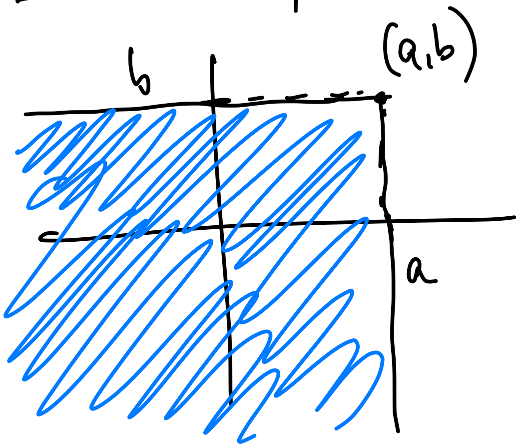
$$P\Big((X,Y)\in R\Big) = \iint_R f_{X,Y}(x,y)\, dA$$

For example, if $R = [a,b] \times [c,d]$, then



$$\iint_R f_{X,Y}(x,y)\, dA = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

<u>Note</u>: May need to consider infinite regions:



$$P\big(X \leq a, Y \leq b\big) = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{X,Y}(x,y)\, dy\, dx$$

Marginals: $P(a \le X \le b) = \int_a^b \left[\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy\right] dx$

$= \int_a^b f_X(x)\,dx.$

$f_X(x)$ is a function of x alone! (prob. density function of X)

$$P(c \le Y \le d) = \int_c^d \left(\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\right) dy = \int_c^d f_Y(y)\,dy.$$

$f_Y(y)$ is a function of y alone (prob. density fct. of Y)

Important consequences.

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy \qquad \text{p.d.f. of } X$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx \qquad \text{p.d.f. of } Y$$

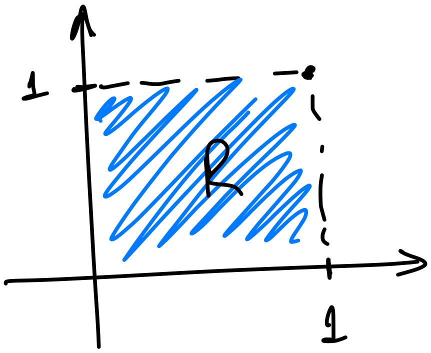
Cumulative Probability: $F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du$

$P(X \le x, Y \le y)$ $\parallel$

(Fund. Thm. of Calc.:) $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}}{\partial x\,\partial y}(x,y)$

Example: Suppose X and Y are jointly distributed with p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{7}(x+y)^2 & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$



a) Check that $f_{X,Y}(x,y)$ is a p.d.f.

b) Compute $P\left(X<\frac{1}{2}, Y<\frac{3}{4}\right)$.

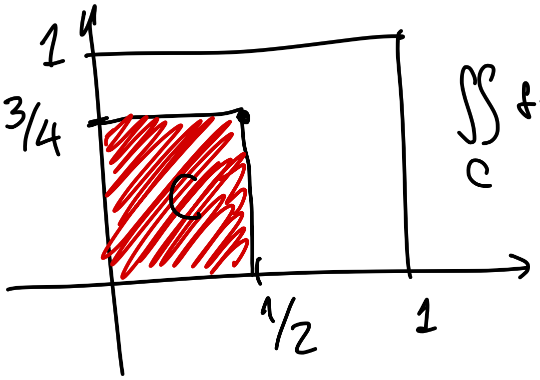
c) Compute $E(X)$.

a) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{6}{7}\underbrace{(x+y)^2}_{x^2+2xy+y^2}\,dx\,dy$$

$$= \frac{6}{7}\int_0^1 \left(\frac{x^3}{3} + x^2y + y^2x\right)\Big|_0^1 dy = \frac{6}{7}\int_0^1 \frac{1}{3} + y + y^2\,dy$$

$$= \frac{6}{7}\left(\frac{y}{3} + \frac{y^2}{2} + \frac{y^3}{3}\right)\Big|_0^1 = \frac{6}{7}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{3}\right) = \frac{6}{7}\cdot\frac{7}{6} = \boxed{1.}$$

b) $$P\left(X<\frac{1}{2}, Y<\frac{3}{4}\right) = \int_0^{3/4}\int_0^{1/2} \frac{6}{7}(x+y)^2\,dx\,dy =$$



$$\| $$

$$\iint_C f_{XY}(x,y)\,dA$$

$$= \frac{6}{7}\int_0^{3/4}\left(\frac{x^3}{3} + x^2y + y^2x\right)\Big|_0^{1/2} dy$$

$$= \frac{6}{7}\int_0^{3/4} \frac{1}{24} + \frac{1}{4}y + \frac{1}{2}y^2\,dy$$

$$= \frac{6}{7}\left(\frac{1}{24}y + \frac{1}{4}\frac{y^2}{2} + \frac{1}{2}\frac{y^3}{3}\right)\Bigg|_0^{3/4}$$

$$= \frac{\cancel{6}}{7}\left(\frac{3}{4\cdot \cancel{24}_{4}} + \frac{1}{\cancel{8}_{4}}\frac{9\cdot 3}{16} + \frac{1}{\cancel{6}}\cdot\frac{27}{64}\right) = \frac{1}{7}\left(\frac{3}{16} + \frac{27}{64} + \frac{27}{64}\right)$$

$$= \frac{1}{7}\,\frac{6+27}{32} = \boxed{\frac{33}{224}}$$

c) $$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-\infty}^{+\infty} x \underbrace{\left(\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy\right)}_{= f_X(x)} dx$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x\,\frac{6}{7}(x+y)^2\,dx\,dy$$

$$= \frac{6}{7}\int_0^1\int_0^1 x^3 + 2x^2y + xy^2\,dx\,dy$$

$$= \frac{6}{7}\int_0^1\left(\frac{x^4}{4} + 2\frac{x^3}{3}y + \frac{x^2}{2}y^2\right)\Bigg|_0^1 dy$$

$$= \frac{6}{7}\int_0^1 \frac{1}{4} + \frac{2y}{3} + \frac{1}{2}y^2\, dy$$

$$= \frac{6}{7}\left(\frac{y}{4} + \frac{y^2}{3} + \frac{y^3}{6}\right)\Big|_0^1 = \frac{6}{7}\left(\frac{1}{4} + \frac{1}{3} + \frac{1}{6}\right) = \boxed{\frac{9}{14}}$$

<u>Bonus:</u> c) done in another way: Find pdf of X first

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^1 \frac{6}{7}(X+Y)^2\, dy$$

$$= \frac{6}{7}\int_0^1 x^2 + 2xy + y^2\, dy = \frac{6}{7}\left(x^2y + xy^2 + \frac{y^3}{3}\right)\Big|_0^1$$

$$= \frac{6}{7}\left(X^2 + X + \frac{1}{3}\right)$$

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^1 x\,\frac{6}{7}\left(x^2 + x + \frac{1}{3}\right)dx$$

$$= \frac{6}{7}\int_0^1 x^3 + x^2 + \frac{x}{3}\,dx = \frac{6}{7}\left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{6}\right)\Big|_0^1$$

$$= \frac{6}{7}\left(\frac{1}{4} + \frac{1}{3} + \frac{1}{6}\right) = \boxed{\frac{9}{14}}$$