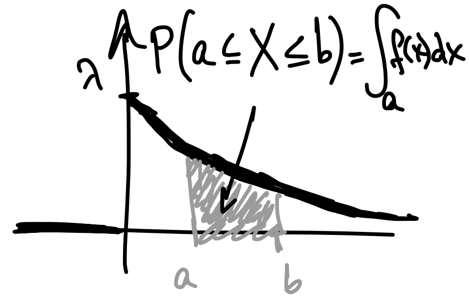
Exponential Random Variables:

Def: A cont. random variable X is an exponential rand. var. if its pdf is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



Verify that $\int_{-\infty}^{+\infty} f(x)\,dx = 1$:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \lambda e^{-\lambda x}\,dx = \lim_{b \to \infty} \int_0^b \lambda e^{-\lambda x}\,dx$$

$$= \lim_{b\to\infty} \cancel{\lambda}\left(\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right)\Bigg|_0^b = \lim_{b\to\infty} -\underbrace{e^{-\lambda b}}_{0} + 1 = 1.$$

Expected Value

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^{+\infty} \lambda x e^{-\lambda x}\,dx = \lim_{b\to\infty} \lambda \int_0^b x e^{-\lambda x}\,dx$$

$$= \lim_{b\to\infty} \lambda \left(\frac{x e^{-\lambda x}}{-\lambda}\Bigg|_0^b - \int_0^b \frac{e^{-\lambda x}}{-\lambda}\,dx \right)$$

$$= \lim_{b\to\infty} -be^{-\lambda b} + 0 + \frac{e^{-\lambda x}}{-\lambda}\Big|_0^b$$

$$= \lim_{b\to\infty} -\underbrace{be^{-\lambda b}}_{\to 0} - \frac{1}{\lambda}\left(\underbrace{e^{-\lambda b}}_{\to 0} - 1\right) = \boxed{\frac{1}{\lambda}}$$

(L'Hospital:
$$\lim_{b\to\infty} -\frac{b}{e^{\lambda b}} = \lim_{b\to\infty} -\frac{1}{\lambda e^{\lambda b}} = 0$$
)

<u>Variance</u>: $Var(X) = E(X^2) - E(X)^2$, $X \sim Exponential(\lambda)$

(Upshot from above computation:
$$\int \lambda x e^{-\lambda x}\,dx = -xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}$$
)

$$E(X^2) = \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\,dx = \lim_{b\to\infty} \lambda \int_0^b x^2 e^{-\lambda x}\,dx =$$

$$= \lim_{b\to\infty} \cancel{\lambda}\left(\frac{x^2 e^{-\lambda x}}{-\cancel{\lambda}}\Big|_0^b - \int_0^b \frac{2xe^{-\lambda x}}{-\cancel{\lambda}}\,dx\right)$$

$$= \lim_{b\to\infty} -b^2 e^{-\lambda b} + 0 + \frac{2}{\lambda}\left(-xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right)\Big|_0^b$$

$$= \lim_{b\to\infty} -b^2 e^{-\lambda b} - \frac{2}{\lambda}\left(b e^{-\lambda b} + \frac{e^{-\lambda b}}{\lambda} - \frac{1}{\lambda}\right)$$

$$= \lim_{b\to\infty} -\underset{\downarrow\, 0}{e^{-\lambda b}}\left(\underbrace{b^2 + \frac{2b}{\lambda} + \frac{2}{\lambda^2}}_{\downarrow\, +\infty}\right) + \frac{2}{\lambda^2} \underset{\text{L'Hospital}}{=} \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2-1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

$$\boxed{\begin{aligned} &\underline{\text{Summary:}} \quad E(X) = \frac{1}{\lambda} \\ &\qquad\qquad \mathrm{Var}(X) = \frac{1}{\lambda^2} \end{aligned}}$$

An important (and distinguishing) property of exponential random variables: memorylessness!

$X \sim \text{Exponential}(\lambda)$

$$P(X \le a) = \int_0^a \lambda e^{-\lambda x}\,dx = \lambda \frac{e^{-\lambda x}}{-\lambda}\Big|_0^a = -e^{-\lambda a} + 1$$

$$= 1 - e^{-\lambda a}$$

$$P(X > a) = 1 - P(X \le a) = 1 - \left(1 - e^{-\lambda a}\right) = e^{-\lambda a}$$

$$P(X > a+b \mid X > a) = \frac{P(X > a+b \text{ AND } X > a)}{P(X > a)}$$

$a, b > 0$

$$= \frac{P(X > a+b)}{P(X > a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= \frac{\cancel{e^{-\lambda a}}\, e^{-\lambda b}}{\cancel{e^{-\lambda a}}} = P(X > b)$$

In other words:

$$P(X > a+b) = P(X > a)P(X > b)$$

"Waiting time" is typically modelled by exp. rand. vr.

In more concrete terms: if X = time until some event happens, and you already waited 30 min, then the probability that you will have to wait at least another 10 min is the same as the (unconditional) probability of having to wait at least 10 min from the beginning.

In other words, the fact (condition) that you already waited 30 min "means nothing at all"!

$$\underbrace{P(X > 40}_{\text{Prob. of it not happening in next 10 min}} \mid \underbrace{X > 30}_{\text{given that you already waited 30 min}}) \text{ IS THE SAME AS } \underbrace{P(X > 10)}_{\text{Prob of it not happening in the "first" 10 min.}}.$$

Exercise: Suppose the time of use it takes until a smartphone fails is modelled by an exponential random variable. The average time until failure is 1000 hours. What is the probability that the smartphone

a) fails in the first 10 hours?

b) doesn't fail in the first 1000 hours?

c) never fails?

X = time until failure [hours]. $\sim \text{Exponential}(\lambda)$

$$E(X) = 1000 \Rightarrow \lambda = \frac{1}{1000} \qquad f(x) = \frac{1}{1000} e^{-\frac{1}{1000}x}$$

a) $$P(X < 10) = \int_0^{10} \frac{1}{1000} e^{-\frac{x}{1000}} dx = \frac{1}{\cancel{1000}} \left(\frac{e^{-\frac{x}{1000}}}{-\frac{1}{\cancel{1000}}} \right) \Bigg|_0^{10}$$

$$= -e^{-\frac{1}{100}} + 1 = 1 - e^{-\frac{1}{100}} \cong 0.00995$$

$(\cong 0.9\%)$

b) $$P(X \geq 1000) = \int_{1000}^{+\infty} \frac{1}{1000} e^{-\frac{x}{1000}} dx$$

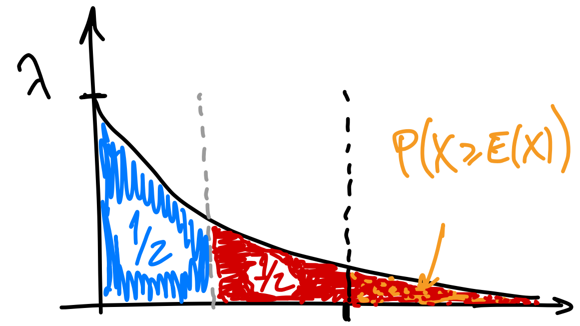
$$= 1 - \int_0^{1000} \frac{1}{1000} e^{-\frac{x}{1000}} dx$$

$$= e^{-\frac{1}{1000} \cdot 1000} = e^{-1} \cong 0.3678 \ (\cong 36.78\%)$$

c) $P(X=+\infty) = \int_{+\infty}^{+\infty} \frac{1}{1000} e^{-\frac{x}{1000}}\, dx = 0$

Obs: The median of $X \sim \text{Exponential}(\lambda)$ is

$$\frac{\ln 2}{\lambda} < \frac{1}{\lambda} = E(X)$$



As seen in the above example:

$$P(X \geq E(X)) < \frac{1}{2}$$

$$P\left(X < \frac{\ln 2}{\lambda}\right) = P\left(X > \frac{\ln 2}{\lambda}\right) = \frac{1}{2}$$

PDF of a function of a random variable

Example: $X \sim$ Uniform on $(0,1)$

How is $Y = X^n$ distributed?

Definition: Cumulative probability

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt$$

Note: $\frac{d}{dx} F_X(x) \overset{\text{F.T.C.}}{=} f_X(x)$.

$$F_Y(y) = P(Y \le y) = P(X^n \le y)$$

$$= P(X \le y^{1/n}) = F_X(y^{1/n}) = y^{1/n}$$

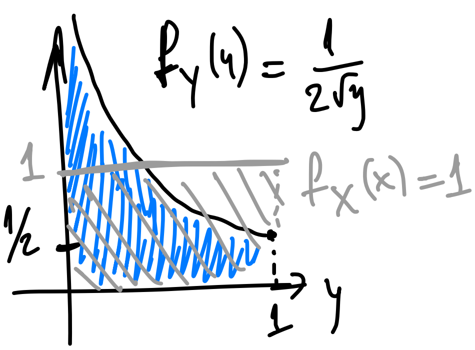
$X \sim$ Uniform

$\left(\begin{array}{l} f_X(x) \equiv 1 \text{ hence} \\ F_X(x) = x \end{array} \right)$

$$\frac{d}{dy} F_Y(y) = \frac{d}{dy} y^{1/n}$$

$$f_Y(y) = \frac{1}{n} y^{\frac{1}{n}-1}$$

If e.g., $n=2$, then $f_Y(y) = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}$



Note: $\int_0^1 \frac{1}{2\sqrt{y}}\, dy = 1.$

Thm: If X is a cont. rand. var. w/ pdf $f_X(x)$; and $g(x)$ is a monotonic function and differentiable then the random variable $Y = g(X)$ has p.d.f

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise.} \end{cases}$$

Pf:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$

$$\left(\begin{array}{l}\text{Say} \\ g \text{ is increasing}\end{array}\right) \quad = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

Differentiate w.r.t. y in both sides:

$$f_Y(y) = F_X'(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y).$$

Similarly if g is decreasing. $\square$