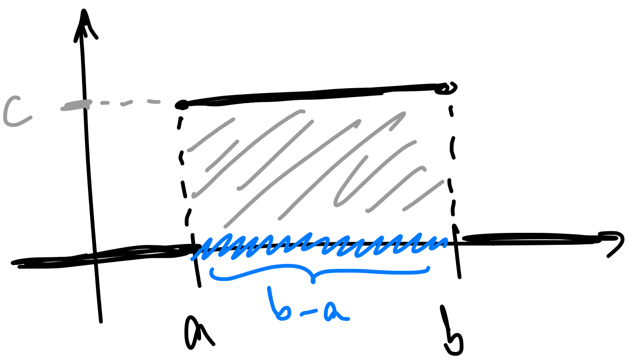
Uniform random variable: A cont. random variable X with values in (a,b) is uniform if its p.d.f is



$$\boxed{f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}}$$

Area $= c \cdot (b-a) = 1$

$\Rightarrow c = \frac{1}{b-a}$

Remember:

$$P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx$$

Expected Value:

$$E(X) = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{b^2 - a^2}{2(b-a)} =$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2} \leftarrow$$

Average of a and b, i.e., the midpoint of the interval (a,b).

Variance:

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx =$$

$$= \frac{1}{b-a}\;\frac{x^3}{3}\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{(a^2+ab+b^2)\cdot 4}{3\cdot 4} - \left(\frac{a+b}{2}\right)^2\frac{3}{3}$$

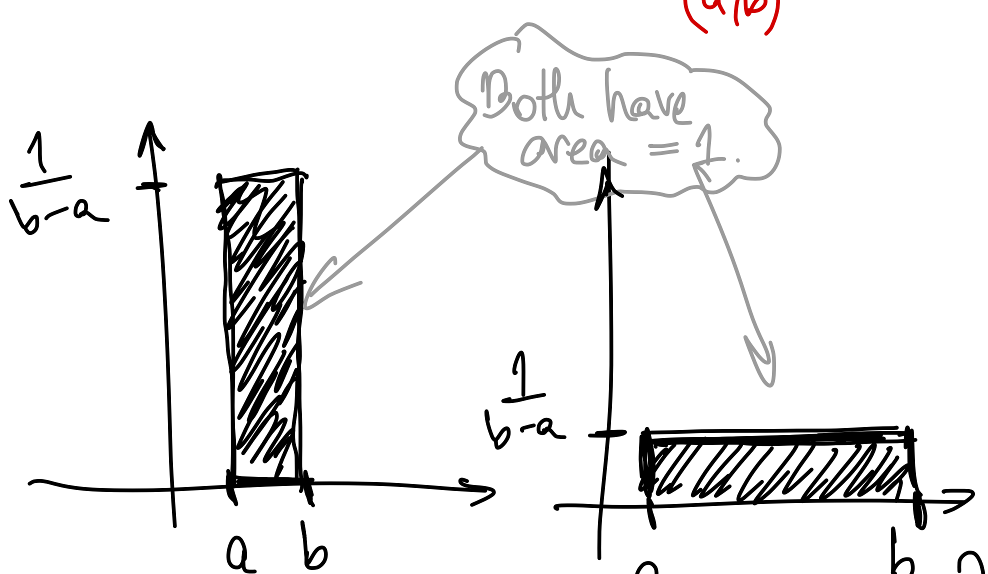
$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \boxed{\frac{(b-a)^2}{12}}$$

$(b-a)$ is the length of the interval (a,b)

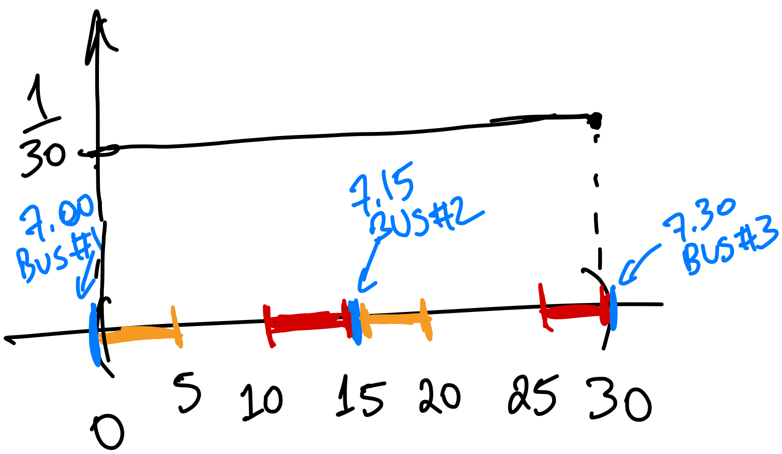
$$E(X) = \frac{a+b}{2}$$

$$Var(X) = \frac{(b-a)^2}{12}$$



Example: Buses leave every 15 min starting at 7am, i.e., 7.00, 7.15, 7.30, ...

You arrive at the bus stop between 7.00 and 7.30, uniformly distributed. What is the prob. that you have to wait:



a) < 5 min?

7.10 - 7.15 or 7.25 - 7.30

b) > 10 min?

7.00 - 7.05 or 7.15 - 7.20

X = min. past 7.00 when you arrive to bus stop

X ~ Uniform on (0, 30)

a) $P(10 \le X \le 15) + P(25 \le X \le 30) =$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx = \frac{1}{30}(5+5)$$

$$\boxed{= \frac{1}{3}}$$

b) $P(0 \le X \le 5) + P(15 \le X \le 20) =$

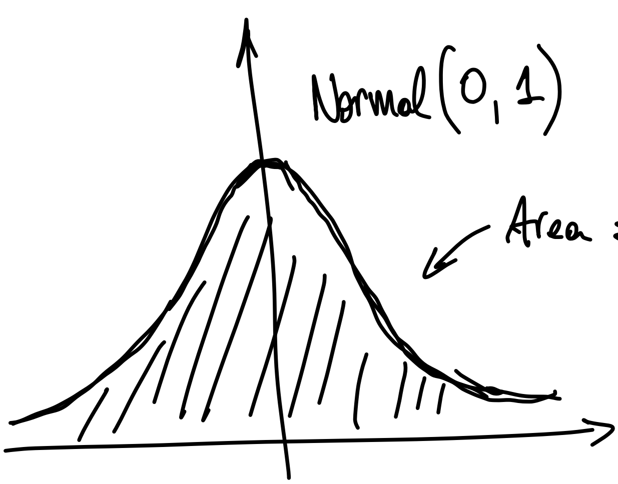
$$= \int_0^5 \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx = \frac{1}{30}(5+5)$$

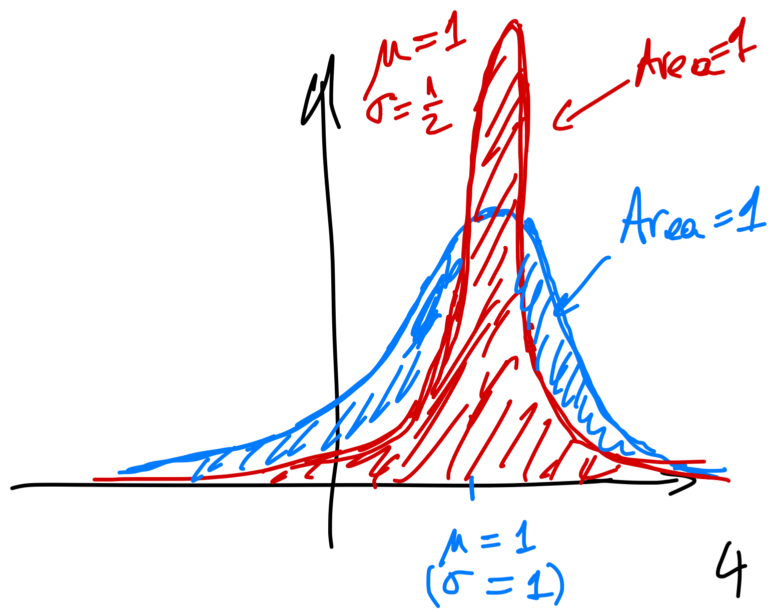
$$= \boxed{\frac{1}{3}}$$

Normal Random Variables. A cont. random variable X is *normal* if it takes values on $(-\infty, +\infty)$ and its p.d.f is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{if } x \in (-\infty, +\infty).$$

$X \sim \text{Normal}(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, then X is called *standard* normal random variable

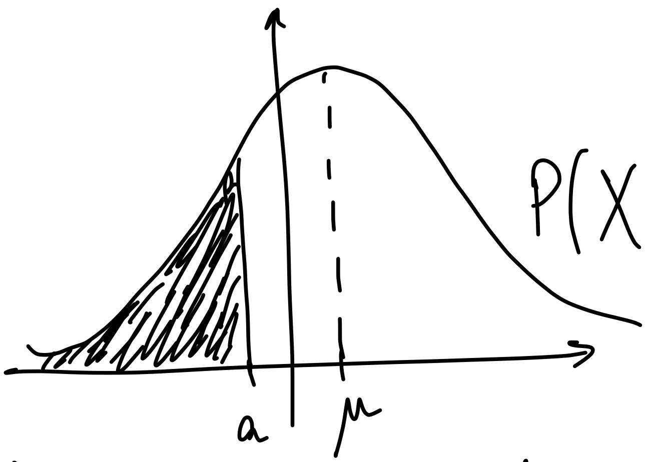




$\mu = E(X)$ mean, expected value

$\sigma^2 = Var(X)$ Variance

⚠ <u>Note</u>: No computations will be required using the normal distribution



$$P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$$ ← This is not computed directly. Use table!

⚠ Normal distr. "appear a lot" in real life, as explained by the <u>Central Limit Theorem</u> ← Later in the semester!

<u>Verifying that</u> $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ <u>is a p.d.f.</u>:

Need to show: $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$

Change of variable $y = \frac{x-\mu}{\sigma}$ brings us to $\left(dy = \frac{dx}{\sigma}\right)$:

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-y^2/2}\,dy$$

Need to show $\int_{-\infty}^{+\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$.

Let $I = \int_{-\infty}^{+\infty} e^{-x^2/2}\,dx$. By Fubini Thm:

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2/2}\,dy\right) \overset{\text{WTS}}{=} 2\pi$$

$$\overset{\text{Fubini}}{=} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy$$

$$\overset{\text{Polar}}{=} \int_0^{2\pi}\int_0^{+\infty} e^{-r^2/2}\,r\,dr\,d\theta = 2\pi\int_0^{+\infty} r e^{-r^2/2}\,dr$$

$$\begin{cases} dx\,dy = r\,dr\,d\theta \\ x^2+y^2 = r^2 \end{cases}$$

$$= -2\pi\left(e^{-r^2/2}\right)\Big|_0^{+\infty} = -2\pi(0-1)$$

$$= 2\pi$$

$\Rightarrow I^2 = 2\pi \Longrightarrow I = \sqrt{2\pi}$ as desired.