Continuous Random Variables

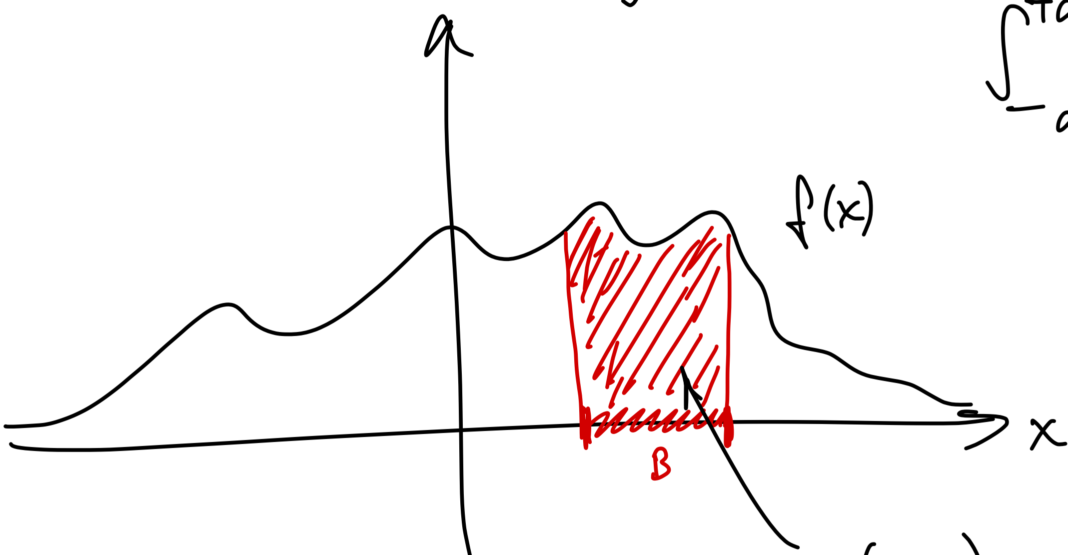
Def: X is a continuous random variable if there is a nonnegative function $f: \mathbb{R} \to [0, \infty)$ such that

$$P(X \in B) = \int_B f(x)\,dx$$

probability density function

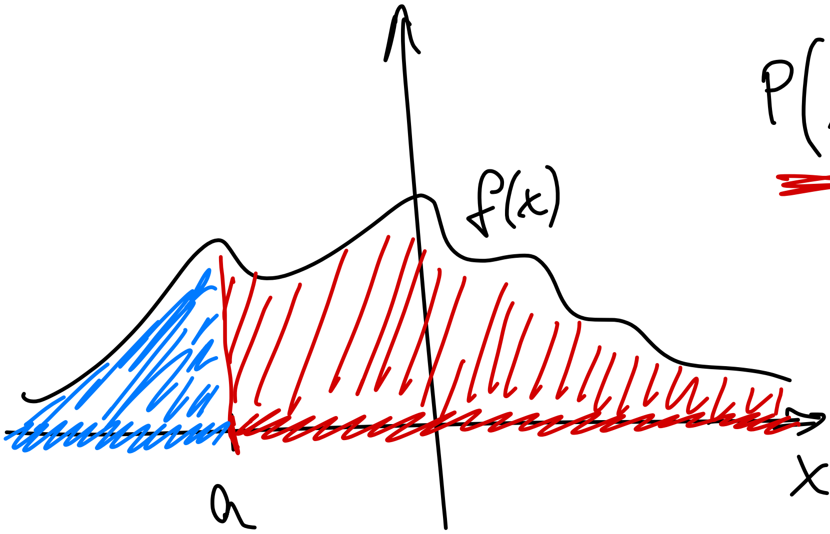
where $B \subset \mathbb{R}$ is any (measurable) subset of $\mathbb{R}$, and

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1.$$



$$P(X \in B) = \frac{\int_B f(x)\,dx}{\underbrace{\int_{-\infty}^{+\infty} f(x)\,dx}_{1}} = \int_B f(x)\,dx$$

is the area under the graph of $f(x)$ over B.



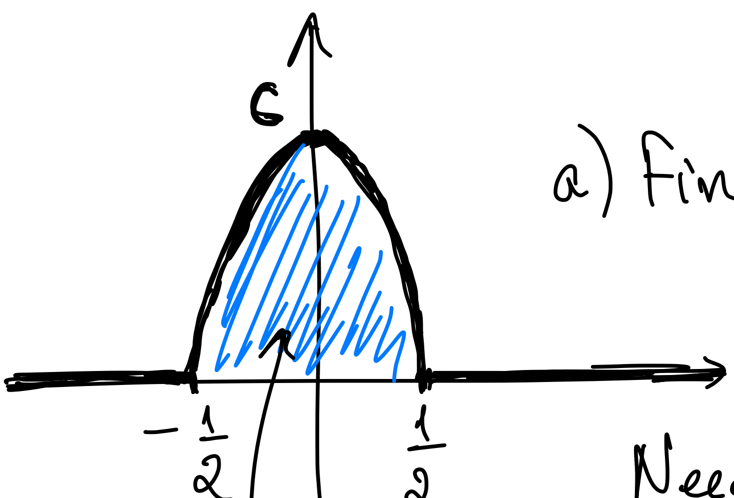
$$\underline{P(X \geq a)} = \int_a^{+\infty} f(x)\,dx$$

$$\underline{P(X \leq a)} = \int_{-\infty}^{a} f(x)\,dx$$

<u>Obs:</u> $P(X=a) = \int_a^a f(x)\,dx = 0.$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx \leftarrow \left(\text{this may be positive}\right)$$

<u>Ex:</u> Suppose X is a random variable w/ prob. density function $f(x) = \begin{cases} C\,(1-4x^2) & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0 & \text{otherwise} \end{cases}$



a) Find $C > 0$ such that the above is a prob. density fct.

<u>Need:</u> $\int_{-\infty}^{+\infty} f(x)\,dx = 1.$

(Area under the graph)

$$\int_{-\infty}^{+\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{-\frac{1}{2}} f(x)\,dx}_{=0} + \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx + \underbrace{\int_{\frac{1}{2}}^{+\infty} f(x)\,dx}_{=0}$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} C\left(1-4x^2\right)dx = C\left(x - \frac{4x^3}{3}\right)\Bigg|_{-\frac{1}{2}}^{+\frac{1}{2}}$$

$$= C\left(\left(\frac{1}{2} - \frac{\cancel{4}}{3}\cdot\frac{1}{\cancel{8}_{2}}\right) - \left(-\frac{1}{2} - \frac{4}{3}\left(-\frac{1}{8}\right)\right)\right)$$

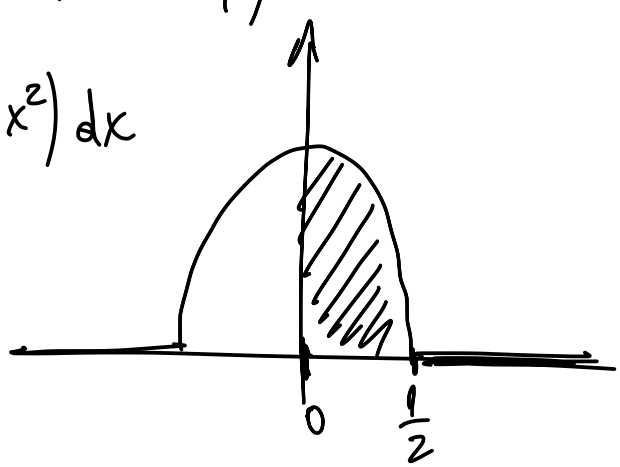
$$= 2C\left(\frac{1}{2} - \frac{1}{6}\right) = 2C\,\frac{3-1}{6} = \frac{2C}{3}.$$

$$\Longrightarrow \quad 1 = \frac{2C}{3} \quad \Longrightarrow \quad \boxed{C = \frac{3}{2}}$$

Now we know $f(x) = \begin{cases} \frac{3}{2}\left(1-4x^2\right) & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

b) Find $P(X \geq 0)$ and $P\left(X \geq \frac{1}{4}\right)$.

$$P(X \geq 0) = \frac{1}{2} = \int_0^{1/2} \frac{3}{2}\left(1-4x^2\right)dx$$

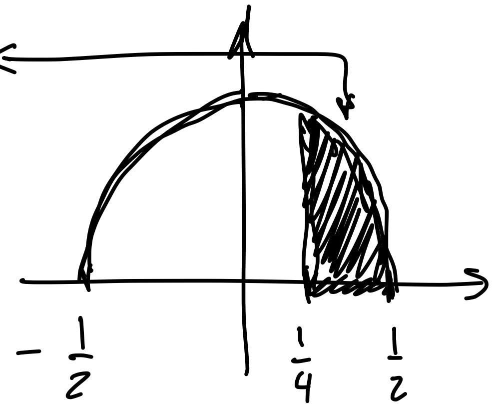


$$P\left(X \geq \frac{1}{4}\right) = \int_{\frac{1}{4}}^{+\frac{1}{2}} \frac{3}{2}\left(1-4x^2\right)dx = \frac{3}{2}\left(x - \frac{4x^3}{3}\right)\Bigg|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{\cancel{4}}{3}\cdot\frac{1}{\cancel{8}} - \left(\frac{1}{4} - \frac{\cancel{4}}{3}\cdot\frac{1}{4^{\cancel{3}2}}\right)\right)$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{6} - \left(\frac{1}{4} - \frac{1}{3.16}\right)\right) = \frac{3}{2}\left(\frac{1}{3} - \frac{12-1}{3.16}\right)$$

$$= \frac{3}{2}\,\frac{16-11}{3.16} = \boxed{\frac{5}{32}}$$



Expected Value

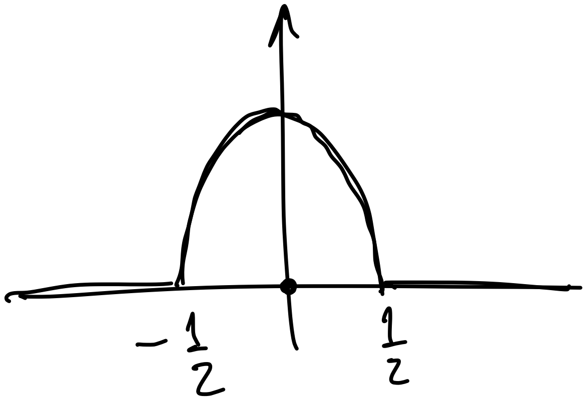
X cont. random variable

$$E(X) = \int_{-\infty}^{+\infty} x\,f(x)\,dx$$

X discrete random variable

$$E(X) = \sum_{x} x\,P(X=x)$$

Previous example:



$$f(x) = \begin{cases} \frac{3}{2}(1-4x^2) & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

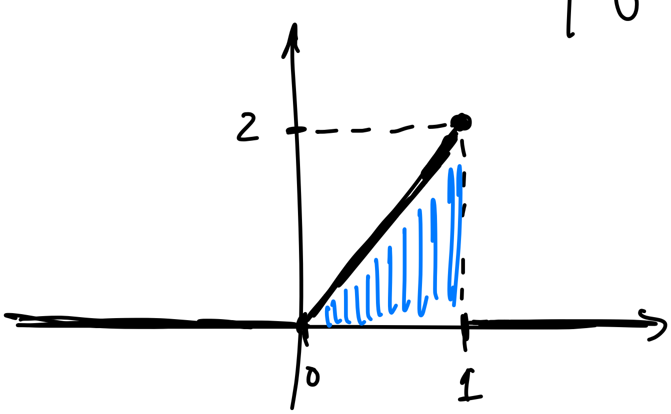
If X has prob. density fct $f(x)$ find $E(X)$?

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_{-\frac{1}{2}}^{+\frac{1}{2}} x\,\frac{3}{2}(1-4x^2)\,dx =$$

$$= \frac{3}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} x - 4x^3\,dx = \frac{3}{2}\left(\frac{x^2}{2} - \frac{\cancel{4}x^4}{\cancel{4}}\right)\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}} =$$

$$= \frac{3}{2}\left(\left(\frac{1}{2}\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^4\right) - \left(\frac{1}{2}\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^4\right)\right) = 0.$$

<u>Ex</u>: Say X is a cont. random variable w/ prob. density function $f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$. Find $E(X)$.



$$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot 2x\,dx$$

$$= 2\,\frac{x^3}{3}\Bigg|_0^1 = \boxed{\frac{2}{3}}$$

Def: If $g(x)$ is a real-valued function, then

$$E(g(X)) = \int_{-\infty}^{+\infty} g(x) f(x)\, dx$$ if X has prob. density function $f(x)$.

Variance of a continuous Random Variable:

$$Var(X) = E\left((X-\mu)^2\right) = E(X^2) - E(X)^2$$

$$\left(\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx \qquad E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx \right)$$

Example: Back to X with prob. density fct

$$f(x) = \begin{cases} \frac{3}{2}(1-4x^2) & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

(We know from before that $E(X)=0$).

$$Var(X) = E(X^2) - E(X)^2 = E(X^2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \underbrace{\frac{3}{2}(1-4x^2)}_{f(x)} dx$$

$$= 3\int_0^{1/2} x^2 - 4x^4 dx = 3\left(\frac{x^3}{3} - \frac{4x^5}{5}\right)\Bigg|_0^{1/2} =$$

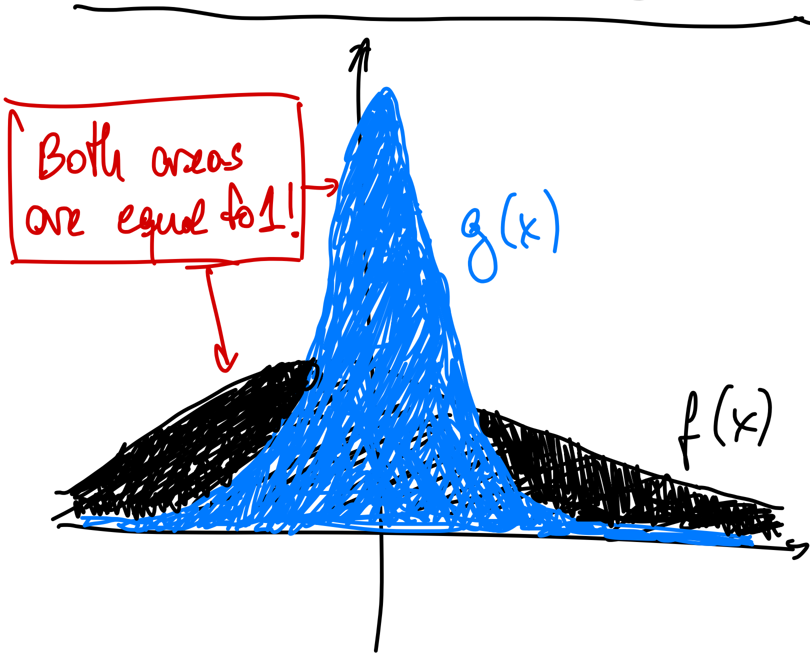
$$= \frac{1}{8} - \frac{\cancel{12}^{\,3}}{5}\frac{1}{\cancel{32}_{\,8}} = \frac{1}{8}\left(1-\frac{3}{5}\right) = \frac{1}{\cancel{8}_{\,4}}\cdot\frac{\cancel{2}}{5} = \boxed{\frac{1}{20}}$$

Example: X is a random variable w/ prob. density function $f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$. (We know $E(X) = \frac{2}{3}$)

$$Var(X) = E(X^2) - E(X)^2 = E(X^2) - \left(\frac{2}{3}\right)^2 = \ldots$$

$$\left[E(X^2) = \int_0^1 x^2 \cdot 2x \, dx = 2\int_0^1 x^3 dx = 2\left.\frac{x^4}{4}\right|_0^1 = \frac{2}{4} = \frac{1}{2} \right]$$

$$\ldots = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \boxed{\frac{1}{18}}$$



Interpretation: Variance is a measure of how concentrated the p.d.f. is

X has p.d.f. $f(x)$

Y has p.d.f. $g(x)$

$$\boxed{Var(X) > Var(Y)}$$

$$SD = \sqrt{Var(X)}.$$