Bernoulli and Binomial Random Variables

Bernoulli Random Variable:

X	0	1
$P(X=x)$	$1-p$	p

$P(X=0) = 1-p$

$P(X=1) = p$

Use this type of random variable to model individual trial, with prob. of success p.

$p(x) = P(X=x)$

$$E(X) = \sum_x x\, p(x) = 0 \cdot \underbrace{p(0)}_{(1-p)} + 1\, \underbrace{p(1)}_{p} = p.$$

$$Var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$$

b/c $E(X^2) = \sum_x x^2 p(x) = 0^2 (1-p) + 1^2 p = p.$

Bernoulli:

$$\boxed{\begin{aligned} E(X) &= p \\ Var(X) &= p(1-p) \end{aligned}}$$

Binomial Random Variable

Use it to model n trials, each of which has prob. p of success. Equivalent to taking the <u>sum</u> of n Bernoulli random variables w/ prob. of success p.

$$\underbrace{\substack{1\\0}}_{1}\quad\underbrace{\substack{1\\0}}_{2}\quad\underbrace{\substack{1\\0}}_{3}\quad\cdots\quad\underbrace{\substack{1\\0}}_{n}\;\longrightarrow\;\boxed{\quad i\quad}\quad\begin{array}{l}i \text{ successes } (1)\\ n-i \text{ failures } (0)\end{array}$$

$$p(i) = \binom{n}{i} p^{i}(1-p)^{n-i} \qquad i = 0, 1, \ldots, n$$

$$X \text{ is Binomial } (\underset{\text{\# of trials}}{n}, \underset{\text{prob. of success}}{p}) \implies X = X_1 + X_2 + \cdots + X_n$$

Bernoulli w/ prob. of success p.

$\boxed{\text{independent}}$

$$E(X) = E(X_1 + X_2 + \cdots + X_n)$$

$$\overset{\text{linearity of } E(\cdot)}{=} E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$= p + p + \cdots + p = n \cdot p$$

$$\mathrm{Var}(X) = \mathrm{Var}(X_1 + \cdots + X_n)$$

$$= \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$$

no $\mathrm{Cov}(X_i, X_j)$ b/c X_i and X_j are independent $\forall i \neq j$

$$= p(1-p) + \cdots + p(1-p)$$

$$= n \cdot p(1-p)$$

Binomial(n, p)

$$\boxed{\begin{aligned} E(X) &= n \cdot p \\ Var(X) &= n p (1-p) \end{aligned}}$$

Example: 5 fair coins are flipped. Find the probability distribution of the number of heads.

$X = \#$ of heads $\qquad n = 5 \qquad p = \frac{1}{2}$

$X \sim \text{Binomial}\left(5, \frac{1}{2}\right)$

$$E(X) = \frac{5}{2} = \sum_x x\, p(x) = 0 \cdot \frac{1}{32} + 1 \cdot \frac{5}{32} + 2\,\frac{10}{32} + 3 \cdot \frac{10}{32} + 4 \cdot \frac{5}{32} + \frac{5}{32}$$

$$= \frac{5 + 20 + 30 + 20 + 5}{32} = \frac{80}{32} = \boxed{\frac{5}{2}}$$

$$P(X=0) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{2^5} = \frac{1}{32}$$

$$P(X=1) = \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 = \frac{5}{32}$$

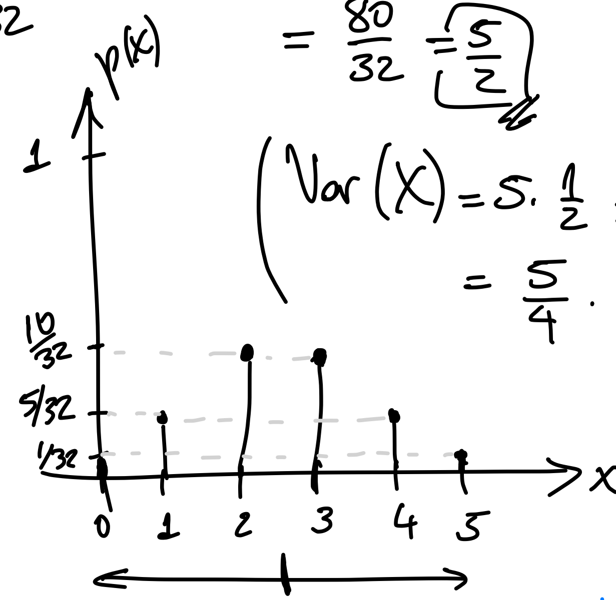
$$P(X=2) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32}$$

$$P(X=3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{10}{32}$$

$$P(X=4) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 = \frac{5}{32}$$

$$P(X=5) = \binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 = \frac{1}{32}$$

$$\left(Var(X) = 5 \cdot \frac{1}{2}\,\frac{1}{2} = \frac{5}{4}. \right)$$



symmetry because $p = 1/2$

Ans:

X	0	1	2	3	4	5
$p(x)$	1/32	5/32	10/32	10/32	5/32	1/32

$$\sum_x P(X=x) = \sum_x p(x) = \frac{1}{32}+\frac{5}{32}+\frac{10}{32}+\frac{10}{32}+\frac{5}{32}+\frac{1}{32}$$
$$= \frac{16}{32}+\frac{16}{32} = \frac{32}{32} = \underline{1}.$$

Example: Population of the Bronx (in 2018): 1.432 million

Current cases of COVID-19 in the Bronx (as of March 29, 4:15pm): 6,250

Q: Suppose you live in a building in the Bronx with 100 people. What is the probability that at least 2 people in your building have COVID-19?

X = # of people with COVID-19.

$$= \underbrace{X_1 + X_2 + \cdots + X_{100}}_{100 \text{ "trials"}}$$

$n = 100$

$p = 0.00436$

$$p = \frac{6{,}250}{1{,}432{,}000} = \frac{25}{5728} \cong 0.00436$$
$(\cong 0.436\%)$

$$P(X \geq 2) = P(X=2) + P(X=3) + \cdots + P(X=100)$$

$$\boxed{P(X=i) = \binom{n}{i} p^i (1-p)^{n-i} = \binom{100}{i} p^i (1-p)^{100-i}}$$

$$= \binom{100}{2} p^2 (1-p)^{98} + \binom{100}{3} p^3 (1-p)^{97} + \cdots + \binom{100}{100} p^{100}$$

$$= \sum_{i=2}^{100} \binom{100}{i} p^i (1-p)^{100-i} \cong 0.0712$$
$(= 7.1\%)$

Simpler computation (equivalent to the above):

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - P(X=0) - P(X=1)$$

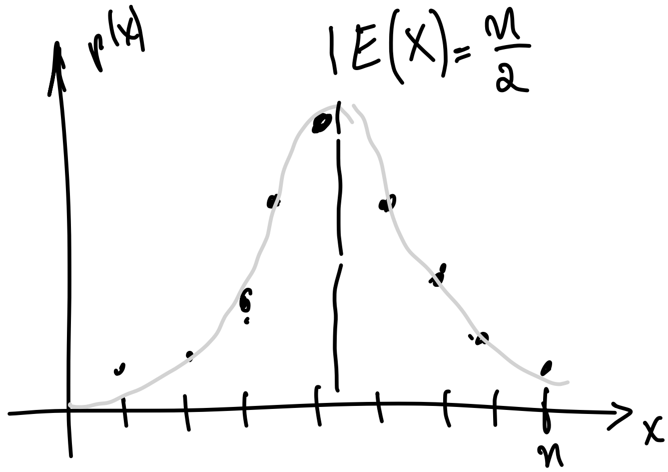
$$= 1 - \binom{100}{0} p^0 (1-p)^{100} - \binom{100}{1} p^1 (1-p)^{99}$$

$$= 1 - (1-p)^{100} - 100p(1-p)^{99}$$

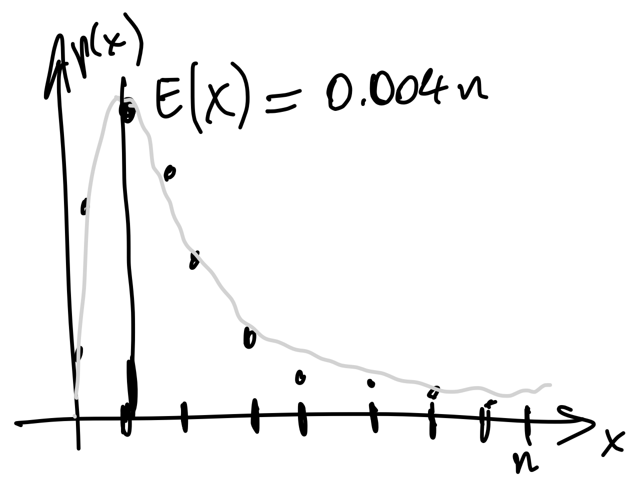
$$\cong 0.071$$

much simpler than the above method.

Comparing prob. mass densities of binomial variables with different prob. of success p:



$\text{Binomial}(n, \frac{1}{2})$



$\text{Binomial}(n, 0.004)$

Check out link on the website for more interactive view of how these distributions change when we change the parameters!