Name: ANSWERS Lehman ID: ______________

MAT 330/681
Final Exam
May 20, 2020

Instructions (PLEASE READ CAREFULLY):

• *Please sign and date the pledge below to comply with the Code of Academic Integrity.*

• *This is an* **OPEN BOOK EXAM**, *meaning that:*
– *YOU ARE ALLOWED to consult any references, notes, and class materials.*
– *YOU ARE NOT ALLOWED to share the contents (in part or in full) of this exam in any forum or tutoring service (such as Chegg, Humbot, Skooli, math.stackexchange, etc.), nor to receive any kind of third-party assistance during the exam. Any violations of the academic integrity code will be fully investigated and penalized as appropriate.*

• *Some questions in this exam require you to generate your OWN UNIQUE INPUT. This is a preventative safeguard to protect students that comply with the academic integrity code. If 2 or more students present the EXACT SAME INPUT to such questions, they will receive ZERO POINTS on that question, even if their answers are correct. Your ability to generate such inputs on your own is part of what is being evaluated.*

• *If anything is unclear, email me at* `r.bettiol@lehman.cuny.edu` *for clarifications.*

• *The amount of time you have to complete the exam is 100 minutes, unless you have a recognized disability.* **You must show all of your work! No credit will be given for unsupported answers.** *Please try to be as organized, objective, and logical as possible in your answers.*

• **Submit your completed exam by 11:59pm on Wednesday, May 20 through the Blackboard Assignment "Final Exam" and be sure to attach clear images of all the pages.**

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

______________________________ ______________
Signature Date

Problem 1 (15 pts): Choose a word with at least 5 letters and write it below:

(Fill in with YOUR OWN UNIQUE INPUT, which must be a word having 5 or more letters. Be creative, since if others have the exact same word you will receive ZERO POINTS; for instance, you could use your last name if it is long/unique enough, or pick an unusual word from a dictionary.)

BETTIOL

a) **(5 pts)** How many different words can be written by reshuffling the letters above? **You must simplify your answer as an explicit integer n.**

$$\frac{7!}{2!} = 7.6.5.4.3 = \boxed{2520}$$

b) **(5 pts)** How many of these words have all vowels appearing together? **You must simplify your answer as an explicit integer n.**

(EIO)BTTL

$$\frac{5!}{2!} \cdot 3! = \boxed{360}$$

c) **(5 pts)** If you pick at random a word that can be written by reshuffling the letters in your word, what is the probability that its vowels **do not** all appear together? **You must simplify your answer as an explicit irreducible fraction a/b.**

$$P = 1 - \frac{360}{2520} = \boxed{\frac{6}{7}}$$

Problem 2 (20 pts): Consider a discrete random variable X whose probability mass function is given by:

(Fill in with YOUR OWN UNIQUE INPUT, making sure these are 3 different real numbers. Be creative enough, since if others have the exact same numbers you will receive ZERO POINTS.)

$$p(x) = \begin{cases} \frac{1}{4} & \text{if } x = \boxed{a}, \\ \frac{1}{2} & \text{if } x = \boxed{b}, \\ \frac{1}{4} & \text{if } x = \boxed{c}. \end{cases}$$

a) **(5 pts)** Compute the probability that X takes a positive value:
You must simplify your answer as an explicit irreducible fraction a/b.

$P(X > 0) =$ $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ depending how many between a, b, c are > 0.

b) **(5 pts)** Compute the expected value of X:
You must simplify your answer as an explicit irreducible fraction a/b.

$E(X) =$ $\frac{a}{4} + \frac{b}{2} + \frac{c}{4}$

c) **(5 pts)** Compute the expected value of X^2:
You must simplify your answer as an explicit irreducible fraction a/b.

$E(X^2) =$ $\frac{a^2}{4} + \frac{b^2}{2} + \frac{c^2}{4}$

d) **(5 pts)** Compute the variance of X:
You must simplify your answer as an explicit irreducible fraction a/b.

$Var(X) =$ $E(X^2) - E(X)^2$

Problem 3 (20 pts): Suppose there are 2 bags containing the following number of red, green, and blue marbles:

(Fill in with YOUR OWN UNIQUE INPUT, making sure these are all positive integer numbers. Be creative enough, since if others have the exact same numbers you will receive ZERO POINTS.)

$$\text{Bag A:} \quad \underbrace{3}_{\text{red marbles}}, \quad \underbrace{a_1}_{\text{green marbles}}, \quad \underbrace{a_2}_{\text{blue marbles}}.$$

$$\text{Bag B:} \quad \underbrace{4}_{\text{red marbles}}, \quad \underbrace{b_1}_{\text{green marbles}}, \quad \underbrace{b_2}_{\text{blue marbles}}.$$

You pick 2 marbles at random from each bag, without seeing their color before picking, and without replacing them.

a) **(5 pts)** What is the probability that you pick exactly 2 red marbles in total?
You must simplify your answer as an explicit irreducible fraction a/b**.**

$A = 3 + a_1 + a_2$

$B = 4 + b_1 + b_2$

$$P_2 = \frac{\binom{3}{2}\binom{A-3}{0}\binom{B-4}{2}\binom{4}{0} + \binom{3}{1}\binom{A-3}{1}\binom{4}{1}\binom{B-4}{1} + \binom{3}{0}\binom{A-3}{2}\binom{4}{2}\binom{B-4}{0}}{\binom{A}{2}\binom{B}{2}}$$

b) **(5 pts)** If you do pick exactly 2 red marbles in total, what is the probability that they both come from bag A?
You must simplify your answer as an explicit irreducible fraction a/b**.**

$$P = \frac{\binom{3}{2}\binom{A-3}{0}\binom{B-4}{2}\binom{4}{0}}{\binom{3}{2}\binom{A-3}{0}\binom{B-4}{2}\binom{4}{0} + \binom{3}{1}\binom{A-3}{1}\binom{4}{1}\binom{B-4}{1} + \binom{3}{0}\binom{A-3}{2}\binom{4}{2}\binom{B-4}{0}}$$

c) **(10 pts)** In general, how many of the 4 marbles you pick are expected to be red?

(Hint: Careful here, this is a somewhat lengthy computation!)
You must simplify your answer as an explicit irreducible fraction a/b.

$$E(X) = P_1 + 2 \cdot P_2 + 3P_3 + 4P_4, \quad \text{where:}$$

$$P_1 = \frac{\binom{3}{1}\binom{A-3}{1}\binom{4}{0}\binom{B-4}{2} + \binom{3}{0}\binom{A-3}{2}\binom{4}{1}\binom{B-4}{1}}{\binom{A}{2}\binom{B}{2}}$$

$$P_3 = \frac{\binom{3}{2}\binom{A-3}{0}\binom{4}{1}\binom{B-4}{1} + \binom{3}{1}\binom{A-3}{1}\binom{4}{2}\binom{B-4}{0}}{\binom{A}{2}\binom{B}{2}}$$

$$P_4 = \frac{\binom{3}{2}\binom{A-3}{0}\binom{4}{2}\binom{B-4}{0}}{\binom{A}{2}\binom{B}{2}}$$

and P_2 as above.

Problem 4 (20 pts): Consider a Binomial random variable X distributed as below:

(Fill with YOUR OWN UNIQUE INPUT, making sure that $n \geq 5$ is an integer, and $0 < p < 1$.)

$$X \sim \text{Binomial}(\underbrace{\quad n \quad}_{n}, \underbrace{\quad p \quad}_{p})$$

a) **(5 pts)** What is the expected value of X?
You must simplify your answer as an explicit irreducible fraction a/b.

$E(X) = n \cdot p$

b) **(5 pts)** What is the standard deviation of X?
You must simplify your answer as an explicit irreducible fraction a/b.

$\sigma(X) = \sqrt{np(1-p)}$

c) **(10 pts)** What is the probability that $X \geq 2$, if you know that $X \geq 1$?
You must simplify your answer as an explicit irreducible fraction a/b.

$$P(X \geq 2 \,|\, X \geq 1) = \frac{P(X \geq 2, X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 2)}{P(X \geq 1)} =$$

$$= \frac{1 - P(X=0) - P(X=1)}{1 - P(X=0)}$$

$$= \frac{1 - \binom{n}{0} p^0 (1-p)^n - \binom{n}{1} p^1 (1-p)^{n-1}}{1 - \binom{n}{0} p^0 (1-p)^n}$$

$$= \boxed{\frac{1 - (1-p)^n - np(1-p)^{n-1}}{1 - (1-p)^n}}$$

Problem 5 (15 pts): Let X and Y be as in Problem 1 of Homework 9.

a) **(5 pts)** Are X and Y independent? Justify.

No: Y is a function of X.

$(Y = e^X)$

b) **(5 pts)** Compute the expected value of Y.

$$E(Y) = \int_1^e y \frac{1}{y}\, dy = \boxed{e-1}$$

$$\left(= \int_0^1 e^x\, dx \right)$$

c) **(5 pts)** Compute the expected value of XY.

$$E(XY) = E(Xe^X) = \int_0^1 xe^x \cdot 1 = e^x(x-1)\Big|_0^1 = \boxed{1}$$

Problem 6 (10 pts): According to a study, average text messages currently exchanged by users in the USA contain approximately 7 words and 1 emoji. Assume that the number of words and emojis in a text message are independent from one another.

a) **(5 pts)** Use Markov's inequality to estimate from above the probability that a text message contains at least 10 words and 2 emojis.
You must simplify your answer as an irreducible fraction a/b.

X = # words, Y = # emojis

$E(X) = 7$, $E(Y) = 1$

$$P(X \geq 10, Y \geq 2) \overset{\text{indep}}{=} P(X \geq 10)\, P(Y \geq 2) \overset{\text{Markov}}{\leq} \frac{E(X)}{10} \cdot \frac{E(Y)}{2}$$

$$= \frac{7}{10} \cdot \frac{1}{2} = \boxed{\frac{7}{20}}$$

b) **(5 pts)** Suppose that the standard deviation of the number of words in a text message is 2. Use Chebyshev's inequality to estimate from below the proportion of text messages that contain between 4 and 10 words.
You must simplify your answer as an irreducible fraction a/b.

$\sigma(X) = 2$

$$P(4 \leq X \leq 10) = 1 - P(|X-7| \geq 3)$$

$$\geq 1 - \frac{2^2}{3^2} = 1 - \frac{4}{9} = \boxed{\frac{5}{9}}$$

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$