6. L. Mirsky, An Introduction to Linear Algebra, Oxford University Press, Oxford, 1955.

7. A. Oppenheim, Inequalities connected with definite Hermitian forms, J. London Math. Soc., 5 (1930) 114–119.

8. I. Schur, Bemerkungen Zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, J. Reine Math., 140 (1911) 1–28.

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COUNTEREXAMPLES TO L'HÔPITAL'S RULE

R. P. BOAS

Department of Mathematics, Northwestern University, Evanston, IL 60201

1. Introduction. I am not, of course, claiming that L'Hôpital's rule is wrong, merely that unless it is both stated and used very carefully it is capable of yielding spurious results. This is not a new observation, but it is often overlooked.

For definiteness, let us consider the version of the rule that says that if f and g are differentiable in an interval (a, b), if

$$\lim_{x\to b-} f(x) = \lim_{x\to b-} g(x) = \infty,$$

and if $g'(x) \neq 0$ *in some interval* (c, b), then

$$\lim_{x\to b-} f'(x)/g'(x) = L$$

implies that

$$\lim_{x\to b-} f(x)/g(x) = L.$$

If $\lim f'(x)/g'(x)$ does not exist, we are not entitled to draw any conclusion about $\lim f(x)/g(x)$. Strictly speaking, if g' has zeros in every left-hand neighborhood of b, then f'/g' is not defined on (a, b), and we ought to say firmly that $\lim f'/g'$ does not exist. There is, however, the insidious possibility that f' and g' contain a common factor: $f'(x) = s(x)\psi(x)$, $g'(x) = s(x)\omega(x)$, where s does not approach a limit and $\lim \psi(x)/\omega(x)$ exists. It is then quite natural to cancel the factor $s(x)$. This is just what we must not do in the present situation: it is quite possible that $\lim \psi(x)/\omega(x)$ exists but $\lim f(x)/g(x)$ does not.

This claim calls for an example. A number of textbooks give one, but it is (as far as I know) always the same example. The aim of this note is both to emphasize the necessity of the condition $g'(x) \neq 0$ and to provide a systematic method of constructing counterexamples when this condition is violated. I consider the case when $b = +\infty$, since the formulas are simpler than when b is finite.

2. A construction. Take a periodic function λ (not a constant) with a bounded derivative, for example $\lambda(x) = \sin x$. Let

$$f(x) = \int_0^x \{\lambda'(t)\}^2\, dt.$$

It is clear that $f(x) \to +\infty$ as $x \to +\infty$. Now choose a function φ such that $\varphi(\lambda(x))$ is bounded and both $\varphi(\lambda(x))$ and $\varphi'(\lambda(x))$ are bounded away from 0. There are many such functions φ; for example,

$$\varphi(x) = e^x \quad \text{or} \quad (x+c)^2 \quad \text{or} \quad 1/(c+x),$$

provided $|\lambda(x)| < c$ and $|\lambda'(x)| < c$. Take $g(x)$ to be $f(x)\varphi(\lambda(x))$. Since $\inf \varphi(\lambda(x)) > 0$, we have $g(x) \to \infty$ as $x \to \infty$.

Now try to apply L'Hôpital's rule to $f(x)/g(x)$. We have to consider $f'(x)/g'(x)$, where

$$f'(x) = \{\lambda'(x)\}^2,$$
$$g'(x) = \{\lambda'(x)\}^2\varphi(\lambda(x)) + f(x)\varphi'(\lambda(x))\lambda'(x).$$

Here $g'(x) = 0$ whenever $\lambda'(x) = 0$, i.e., g' has zeros in every neighborhood of ∞, and consequently we are not entitled to apply L'Hôpital's rule at all. However, this conclusion seems rather pedantic; let us go ahead anyway. If we cancel the factor $\lambda'(x)$, we obtain

$$\frac{f'(x)}{g'(x)} = \frac{\lambda'(x)}{\lambda'(x)\varphi(\lambda(x)) + f(x)\varphi'(\lambda(x))}.$$

Now $\lambda'(x)$ is bounded (by hypothesis), $\lambda'(x)\varphi(\lambda(x))$ is bounded, $\varphi'(\lambda(x))$ is bounded away from 0, but $f(x) \to \infty$, so $f'(x)/g'(x) \to 0$. Yet $f(x)/g(x) = 1/\varphi(\lambda(x))$ does not approach zero, since $\varphi(\lambda(x))$ is bounded!

3. Discussion. What went wrong? If you will study any proof of L'Hôpital's rule, you will find a place where it used (or should have used) the assumption that $g'(x)$ did not change sign infinitely often in a neighborhood of ∞. Our example shows that, at least sometimes, L'Hôpital's rule actually fails when this hypothesis is not satisfied.

The phenomenon just described was discovered more than a century ago by O. Stolz [**1**], [**2**]. His example was $\lambda(x) = \sin x$, $\varphi(x) = e^x$; it has been repeated in all the modern discussions that I have seen. It was wondering whether there *are* any other examples that led to this note.

One can verify that it is the changes of sign of $\lambda'(x)$ that cause the trouble, not the mere presence of zeros of λ'. In other words, if $\lambda'(x) \geqslant 0$, the cancellation process still leads to a correct result, as Stolz pointed out. However, it seems wildly improbable that an example of either kind will occur in practice, especially for limits at a finite point. Differentiable functions with infinitely many changes of sign in a finite interval are rarely encountered outside notes like this one; all the less, functions with infinitely many double zeros.

4. History. Guillaume François Antoine de Lhospital, Marquis de Sainte-Mesme (1651–1704) published (anonymously) in 1691 the world's first textbook on calculus, based on John Bernoulli's lecture notes. He seems to have written his name as above, but it is more familiar as L'Hospital (old French spelling) or L'Hôpital (modern French); I prefer the latter, since it stops students from pronouncing the s (which Larousse's dictionary says is not to be pronounced).

References

1. O. Stolz, Ueber die Grenzwerthe der Quotienten, Math. Ann., 15 (1879) 556–559.
2. ______, Grundzüge der Differential- und Integralrechnung, vol. 1, Teubner, Leipzig, 1893, pp. 72–84.

CONVOLUTIONS OF CAUCHY DISTRIBUTIONS

COLIN R. BLYTH

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada K7L 3N6

Recently in this MONTHLY Dwass [**1**] and Nelson [**3**] have discussed finding the distribution of a sum of two independent Cauchy random variables using the convolution formula with partial