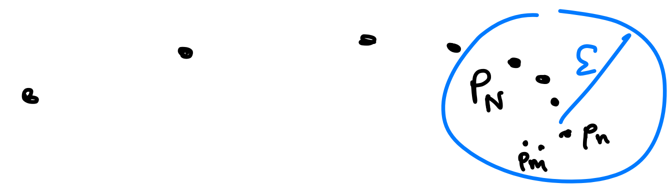
MAT320 Lecture 9 9/28/2020

Def: A sequence $\{p_n\}$ in a metric space (X,d) is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. if $n, m \geq N$, then $d(p_n, p_m) < \varepsilon$.



Using

$$\operatorname{diam} E = \sup \{ d(x,y) : x, y \in E \}$$

Prop: A seq. $\{p_n\}$ is Cauchy if and only if

$$\lim_{N \to \infty} \operatorname{diam} \{ p_n : n \geq N \} = 0.$$

Note: As we will see shortly, a Cauchy seq. may or may not converge (depending on whether the space it is in has the property of being "complete").

Thm: a) If $\overline{E}$ is the closure of E, then

$$\operatorname{diam} \overline{E} = \operatorname{diam} E$$

b) If K_n is a seq. of compact sets in X s.t. $K_n \supset K_{n+1}$, $\forall n \in \mathbb{N}$, and if

$$\lim_{n \to \infty} \operatorname{diam} K_n = 0$$

then $\bigcap_{n \in \mathbb{N}} K_n$ consists of exactly one point.

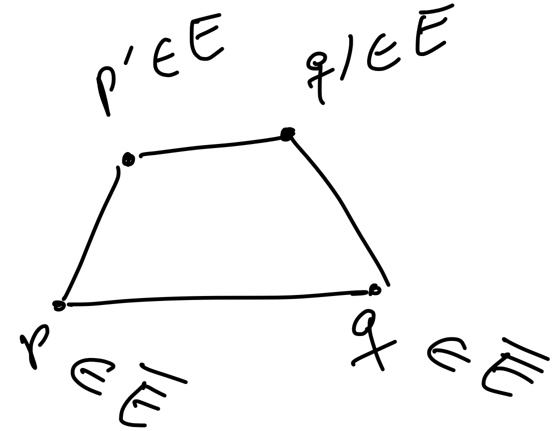
Pf: a) Since $E \subset \overline{E}$, it follows that

$$\operatorname{diam} E \leq \operatorname{diam} \overline{E}$$

Conversely, fix $\varepsilon > 0$ and $p, q \in \overline{E}$. Since $p, q \in \overline{E}$, $\exists p', q' \in E$ s.t. $d(p, p') < \varepsilon$ and $d(q, q') < \varepsilon$. Thus:

$$d(p, q) \leq d(p, p') + d(p', q') + d(q', q)$$

$$< 2\varepsilon + d(p',q')$$

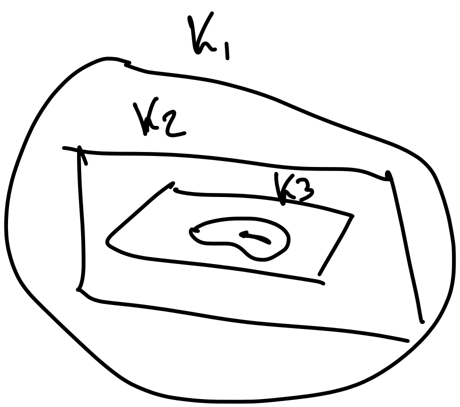


$$\Rightarrow \quad d(p,q) \overset{(*)}{<} 2\varepsilon + \operatorname{diam} E.$$

Since $\varepsilon > 0$ is arbitrary, and can be chosen as small as desired, it follows that:

$$\sup\{d(p,q) : p,q \in \bar{E}\} = \operatorname{diam} \bar{E} \underset{(*)}{\leq} \operatorname{diam} E$$

b) Let $K = \bigcap_{n \in \mathbb{N}} K_n$. By a result proven before (Video 6, Lecture 5), $K \neq \emptyset$. If K contains more than 1 point, then $\operatorname{diam} K > 0$, but this contradicts $\lim_{n\to\infty} \operatorname{diam} K_n = 0$ because $\operatorname{diam} K_n \geq \operatorname{diam} K$.



$\square$

Thm. a) Every convergent sequence is a Cauchy sequence.

b) Every Cauchy seq. in a compact metric space converges.

c) Every Cauchy seq. in $\mathbb{R}^k$ converges.

Pf: a) If $\{p_n\}$ is a convergent seq., say $p_n \to p_\infty$, then $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t., if $n \geq N$, $d(p_n, p_\infty) < \varepsilon/2$. Then if $m, n \geq N$, we have:

$$d(p_n, p_m) \leq d(p_n, p_\infty) + d(p_\infty, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means that $\{p_n\}$ is Cauchy.

b) Let $\{p_n\}$ be a Cauchy seq. in a compact metric space X. Let $E_N = \{p_n : n \geq N\}$. Then, by the above,

$\lim_{N \to \infty} \operatorname{diam} \overline{E_N} = 0$. Moreover, since $\overline{E_N}$ is a closed subset of the compact metric space X, it is also compact. Clearly $E_N \supset E_{N+1}$, $\forall N \in \mathbb{N}$, so also $\overline{E_N} \supset \overline{E_{N+1}}$, $\forall N \in \mathbb{N}$. By the theorem above (part b), it follows that $\bigcap_{N \in \mathbb{N}} \overline{E_N} = \{p_\infty\}$ consists of a single point.

Given $\varepsilon > 0$, since $\operatorname{diam} \overline{E_N} \xrightarrow{N \nearrow \infty} 0$, $\exists N_0 \in \mathbb{N}$ s.t. $\operatorname{diam} \overline{E_N} < \varepsilon$ for $N \geq N_0$. Since $p_\infty \in \overline{E_N}$ we have that $d(p, p_\infty) < \varepsilon$ $\forall p \in \overline{E_N}$, in particular also $d(p, p_\infty) < \varepsilon$, $\forall p \in E_N = \{p_N, p_{N+1}, \dots\}$. This precisely means that $d(p_n, p_\infty) < \varepsilon$ for $n \geq N_0$, i.e., the Cauchy seq. $\{p_n\}$ converges to p_∞.

c) Let $\{p_n\}$ is a Cauchy seq. in $\mathbb{R}^k$. Let E_N be as before, that is, $E_N = \{p_N, p_{N+1}, \dots\}$. For some $N \in \mathbb{N}$, $\operatorname{diam} E_N < 1$. Thus, since

$$\{p_n : n \in \mathbb{N}\} = \underbrace{\{p_1, p_2, \dots, p_{N-1}\}}_{\text{finitely many pts}} \cup \underbrace{E_N}_{\operatorname{diam} < 1}$$

it follows that $\{p_n\}$ is bounded. Therefore, its closure, $\overline{\{p_n : n \in \mathbb{N}\}} \subset \mathbb{R}^k$ is compact. By the previous item (b), it follows that $\{p_n\}$ is convergent. $\square$

<u>Def</u>: A metric space (X,d) is <u>complete</u> if every Cauchy seq in (X,d) is convergent.

By the previous Theorem:

- compact metric spaces are complete
- $\mathbb{R}^k$ is complete.

For example, $(\mathbb{Q}, d)$ is not complete; for instance, the seq. $\{x_n\}$ defined inductively by setting $x_1 = 1$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \in \mathbb{Q}$$

This is a Cauchy seq. in $\mathbb{Q}$ (b/c it is Cauchy in $\mathbb{R}$)

The above x_n converges to $\sqrt{2} \notin \mathbb{Q}$.

Also, $(\mathbb{R} \setminus \mathbb{Q}, d)$ is not complete; for instance, let $x_n = \frac{\sqrt{2}}{n}$, $n \in \mathbb{N}$, and note that $x_n \in \mathbb{R} \setminus \mathbb{Q}$, while

$$x_n \longrightarrow 0 \in \mathbb{Q}.$$

This is a Cauchy seq. in $\mathbb{R} \setminus \mathbb{Q}$ b/c it is in $\mathbb{R}$, or ...

Exercise: Find such an N!

$$\left| \frac{\sqrt{2}}{n} - \frac{\sqrt{2}}{m} \right| = \sqrt{2} \left| \frac{n-m}{nm} \right| < \varepsilon,$$

for all $n, m \geq N$

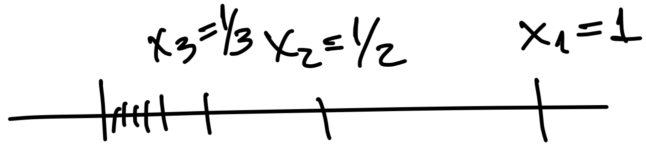
Def: A seq. $\{x_n\}$ of real numbers is

a) <u>monotonic increasing</u> if $x_n \le x_{n+1}$, $\forall n \in \mathbb{N}$

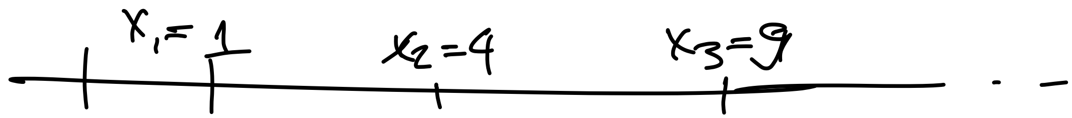
b) <u>monotonic decreasing</u> if $x_n \ge x_{n+1}$, $\forall n \in \mathbb{N}$.

A seq. is monotonic if it is either monotonic increasing or monotonic decreasing.

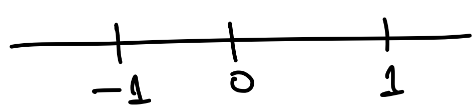
<u>Ex:</u> $x_n = \frac{1}{n}$ monotonic decreasing $\quad \frac{1}{n} > \frac{1}{n+1}$, $\forall n \in \mathbb{N}$



$x_n = n^2$ monotonic increasing $\quad n^2 < (n+1)^2$, $\forall n \in \mathbb{N}$

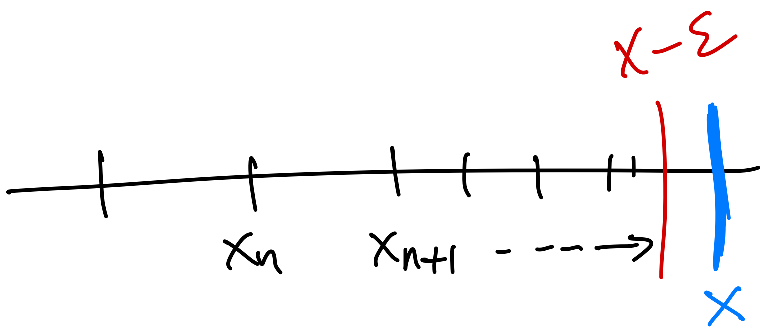


$x_n = (-1)^n$ is <u>not</u> monotonic.



<u>Thm</u>: Suppose $\{x_n\}$ is monotone. Then $\{x_n\}$ is convergent if and only if it is bounded.

<u>Pf</u>: WLOG, say $\{x_n\}$ is monotonic increasing, i.e., $x_n \le x_{n+1}$, $\forall n \in \mathbb{N}$. If $\{x_n\}$ is bounded, then



$$x := \sup \underbrace{\{x_n : n \in \mathbb{N}\}}_{\text{bounded}}$$

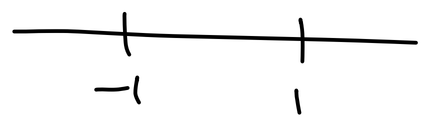
Then x is an upper bound for x_n, i.e., $x_n \le x, \forall n \in \mathbb{N}$.
For $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $x - \varepsilon < x_N \le x$; since otherwise $x - \varepsilon$ would an upper bound smaller than the sup.
Since $x_n \ge x_N$ if $n \ge N$, we have that for all $n \ge N$, $x - \varepsilon < x_n \le x$; this means that

$x_n \to x$, as desired. Converse is always true (see below). $\square$

Remark: Every convergent seq. is bounded.

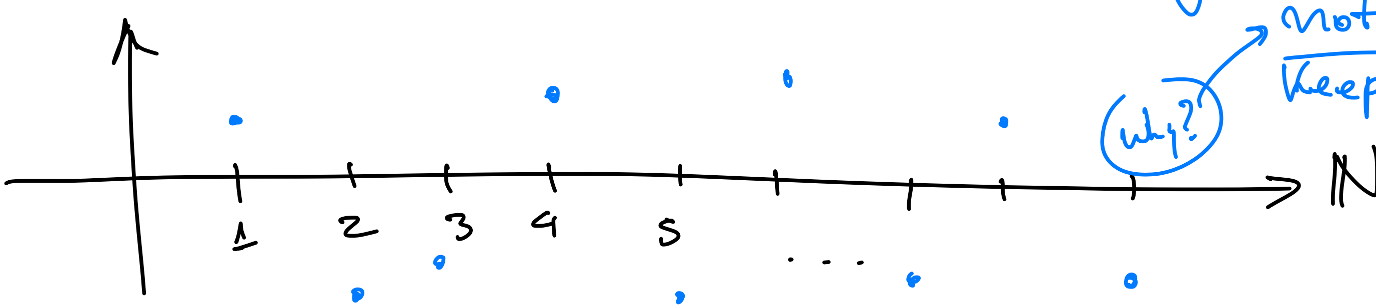
Take $\varepsilon = 1$ in the def. of convergence! (Lecture 8)

But there are bounded seq. that are not convergent... e.g., $x_n = (-1)^n$.



Upper and Lower Limits (lim sup and lim inf)

Intuition: $x : \mathbb{N} \to \mathbb{R}$



E.g.: $x_n = \sin(n)$ does not converge; (why?) it keeps oscillating from -1 to 1, but:

$$\limsup x_n = 1, \quad \liminf x_n = -1, \text{ even though } \nexists \lim x_n$$

Def: Let $\{x_n\}$ be a seq. of real numbers, let

$$E = \{ x \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} : x \text{ is a \underline{subsequential limit of } } \{x_n\} \}$$

i.e. $\exists \{x_{n_k}\}$ a subseq. of $\{x_n\}$ s.t. $x_{n_k} \to x$.

$$\limsup_{n\to\infty} x_n = \sup E \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$$

$$\liminf_{n\to\infty} x_n = \inf E \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$$

(Recall: In Video 6 of Lecture 8 we proved E is closed.)

Note: $E \subset \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ might be <u>unbounded</u>.

Examples:

$$\left.\begin{array}{l} \limsup_{n\to\infty} \sin(n) = 1 \\ \\ \limsup_{n\to\infty} \cos(n) = 1 \end{array}\right\}$$

in both cases, the set of subsequential limits is $E = [-1, 1]$.

$$\liminf_{n\to\infty} \sin(n) = -1 = \liminf_{n\to\infty} \cos(n)$$

$$\left.\begin{array}{l}\limsup_{n\to\infty} n^2 = +\infty \\ \\ \liminf_{n\to\infty} n^2 = +\infty.\end{array}\right\}$$ In this case $E = \{+\infty\} \ldots$

Note: If $\{x_n\}$ converges, then:

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

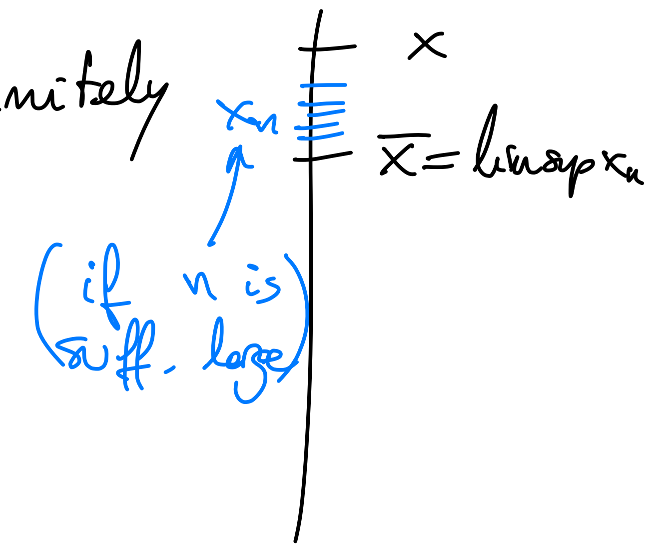
Thm: Let $\{x_n\}$ be a seq. of real numbers, and $E \subset \mathbb{R} \cup \{\pm\infty\}$ be the set of subsequential limits of $\{x_n\}$. Then $\overline{x} = \limsup x_n$ is the only number s.t. $\overline{x} \in E$ and if $x > \overline{x}$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$ then $x_n < x$.

Pf: If $\overline{x} = +\infty$, then E is not bounded from above, i.e. $\exists x_{n_k}$ subseq. s.t. $x_{n_k} \to +\infty$; by def., this means $\overline{x} = +\infty \in E$.

If $\overline{x} \in \mathbb{R}$, then E is bounded from above, hence at least one subseq. limit exists (in $\mathbb{R}$) and $\overline{x} = \sup E \in E$ follows from the fact that E is closed (Video 6 of Lecture 8).

If $\overline{x} = -\infty$, then $E = \{-\infty\}$, so for all $M \in \mathbb{R}$, $x_n > M$ for at most finitely many $n \in \mathbb{N}$, so $x_n \to -\infty$, so again $-\infty \in E$.

Suppose $x > \overline{x}$, and $x_n \geq x$ for infinitely many $n \in \mathbb{N}$. Then $\exists y \in E$ s.t. $y \geq x > \overline{x}$, which contradicts $\overline{x} = \sup E$.



Finally, to prove uniqueness of $\overline{x}$, suppose that $p, q \in \mathbb{R} \cup \{\pm\infty\}$ with the above properties, and $p < q$.

Choose x s.t. $p < x < q$.

Since p satisfies the property that if $x > p$ then $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n < x$, we have $x_n < x$ for $n \geq N$. But then $q \notin E$; contradiction $\square$

<u>Examples:</u> • $x_n = \dfrac{(-1)^n}{1 + \frac{1}{n}}$ has $\limsup_{n \to \infty} x_n = 1$ (n even)

$\liminf_{n \to \infty} x_n = -1$ (n odd)

• $\{x_n : n \in \mathbb{N}\} = \mathbb{Q}$ ← Since $\mathbb{Q}$ is countable, there exists a seq. $\{x_n\}$ s.t. every rational number belongs to $\{x_n\}$.

$E = \mathbb{R} \cup \{\pm\infty\}$ ← <u>Every</u> real number is a subseq. limit. of x_n.

$\liminf_{n \to \infty} x_n = -\infty$, $\limsup_{n \to \infty} x_n = +\infty$.