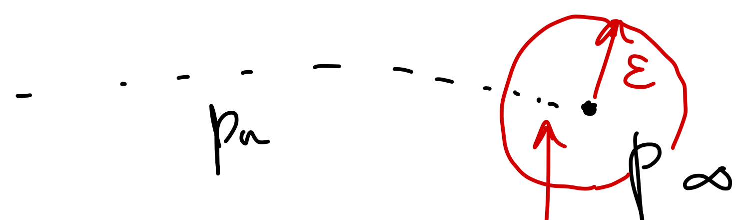
MAT320 Lecture 8 9/23/2020

Sequences

Def: A sequence $\{p_n\}$ in a metric space (X,d) is a function $p: \mathbb{N} \to X$, $p_n = p(n)$. The sequence $\{p_n\}$ is said to converge to $p_\infty \in X$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $d(p_n, p_\infty) < \varepsilon$.



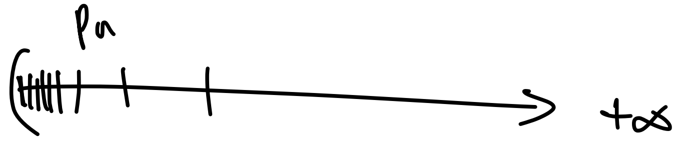
all p_n with $n \geq N$ are ε-close to p_∞.

Notation: $p_n \longrightarrow p_\infty$ in (X,d)

or $\lim\limits_{n \to \infty} p_n = p_\infty$.

Remark: Convergence depends on the ambient space (X,d):
e.g., if $p_n = \frac{1}{n}$, $n \in \mathbb{N}$ then $p_n \longrightarrow 0$ in $(\mathbb{R}, d)$ but

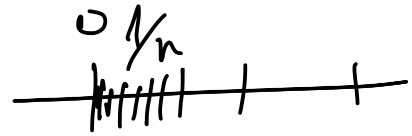
p_n does not converge in $((0,+\infty), d)$ ← is a metric space itself



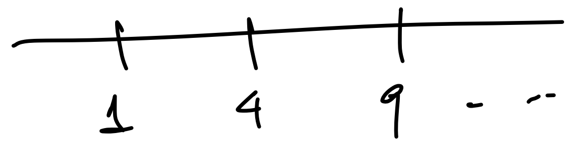
Def: Sequences that do not converge are said to **diverge**.

Ex: Some sequences in $(\mathbb{R}, d)$

(1) $a_n = \frac{1}{n}$, $\quad a_n \longrightarrow 0$.

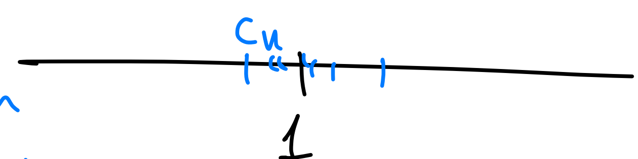


(2) $b_n = n^2$ $\quad b_n$ is divergent.



(3) $c_n = 1 + \frac{(-1)^n}{n}$

$$\frac{(-1)^n}{n} = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$



$c_n \longrightarrow 1$.

"Pf": Want: $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ if $n \geq N$ then $\frac{1}{n} = \left| \cancel{1} + \frac{(-1)^n}{n} - \cancel{1} \right| < \varepsilon$

Pf: $\forall \varepsilon > 0$ by the Archimedean prop. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

If $n \geq N$, then $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$ so:

$$|c_n - 1| = \Big|\underbrace{\Big(1 + \frac{(-1)^n}{n}\Big)}_{c_n} - 1\Big| = \Big|\frac{(-1)^n}{n}\Big| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

$\square$

(e) $d_n = 1$

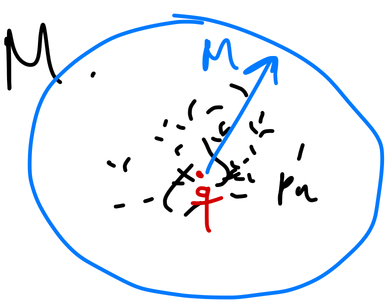
$d_n \to 1$.

Pf: $\forall \varepsilon > 0$, let $N = 1$. Then $\forall n \geq N$ i.e. $\forall n \in \mathbb{N}$,

$$|d_n - 1| = |1 - 1| = 0 < \varepsilon.$$

$\square$

Def: A sequence $\{p_n\}$ in the metric space (X, d) is bounded if the set $\{p_n : n \in \mathbb{N}\}$ is bounded. In other words, $\exists q \in X$ and $M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $d(p_n, q) \leq M$.



Thm: Let $\{p_n\}$ be a seq. in the metric space (X,d).

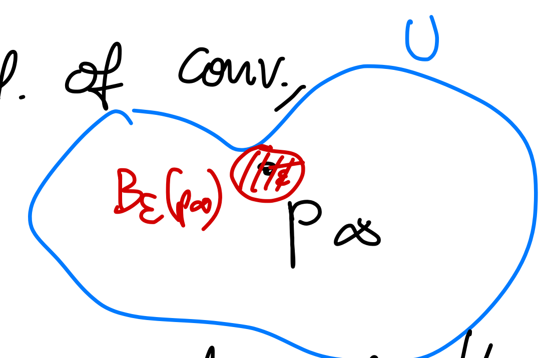
(i) $p_n \to p_\infty$ if and only if $\forall U \ni p_\infty$ open neighbd., U contains infinitely many p_n's.

(ii) if $p_n \to p_\infty$ and $p_n \to p'_\infty$, then $p_\infty = p'_\infty$.

(iii) if $p_n \to p_\infty$, then $\{p_n\}$ is bounded

(iv) if $E \subset X$ and $p \in X$ is a limit point of E, then there is a seq $\{p_n\}$ in E s.t. $p_n \to p$.

Pf: (i) $p_n \to p_\infty$. Let $U \ni p_\infty$ be an open neighbd, then $\exists \varepsilon > 0$ s.t. $B_\varepsilon(p_\infty) \subset U$. Then, by def. of conv., $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ then



$d(p_n, p_\infty) < \varepsilon \implies p_n \in B_\varepsilon(p_\infty) \subset U$.

So U contains infinitely many p_n's (namely, all those for which $n \geq N$.)

Conversely, suppose $\forall V \ni p_\infty$ open neighbhd, U contains infinitely many p_n's. Given $\varepsilon > 0$, let $U = B_\varepsilon(p_\infty)$. So $\exists N \in \mathbb{N}$ s.t. if $n \geq N$ then $d(p_n, p_\infty) < \varepsilon$; which means that $p_n \to p_\infty$.

(ii) Suppose $p_n \to p_\infty$ and $p_n \to p_\infty'$. If $p_\infty \neq p_\infty'$, then $d(p_\infty, p_\infty') > 0$. Let $\varepsilon = \frac{1}{2} d(p_\infty, p_\infty') > 0$. Since $p_n \to p_\infty$, $\exists N_1 \in \mathbb{N}$ s.t. if $n \geq N_1$ then $d(p_n, p_\infty) < \varepsilon$.

Since $p_n \to p_\infty'$, $\exists N_2 \in \mathbb{N}$ s.t. if $n \geq N_2$ then $d(p_n, p_\infty') < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$,

$$0 < 2\varepsilon = d(p_\infty, p_\infty') \leq d(p_\infty, p_n) + d(p_n, p_\infty') < \varepsilon + \varepsilon = 2\varepsilon$$

The above contradiction implies $p_\infty = p_\infty'$.

(iii) If $p_n \to p_\infty$, take $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ s.t.

if $n \geq N$, $d(p_n, p_\infty) < 1$

Let $r = \max\{1, d(p_\infty, p_1), d(p_\infty, p_2), \ldots, d(p_\infty, p_N)\}$.

Then $d(p_n, p_\infty) \leq r$, $\forall n \in \mathbb{N}$; so $\{p_n\}$ is bounded.

(iv) Since $p \in X$ is a limit point of E, $\forall U \ni p$ open neighbhd, $\exists q \in (E \setminus \{p\}) \cap U$. Applying this with $U = B_{\frac{1}{n}}(p)$, and setting $p_n := q$, we have a seq. $\{p_n\}$ s.t. $d(p_n, p) < \frac{1}{n}$. Given $\varepsilon > 0$, by Archimedean prop, $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$. Thus if $n \geq N$, then

$d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$; so $p_n \to p$.

$\square$

Thm. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real (or complex) numbers, s.t. $a_n \to a$ and $b_n \to b$. Then:

(i) $a_n + b_n \to a + b$

(ii) $a_n \cdot b_n \to a \cdot b$

(iii) $c \cdot a_n \to c \cdot a$ and $a_n + c \to a + c$, $\forall c$

(iv) $\frac{1}{a_n} \to \frac{1}{a}$ if $a_n \neq 0$, $\forall n \in \mathbb{N}$ and $a \neq 0$.

Pf: (i) Since $a_n \to a$, $\forall \varepsilon > 0$ $\exists N_1 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow |a_n - a| < \varepsilon/2$

Since $b_n \to b$, $\forall \varepsilon > 0$ $\exists N_2 \in \mathbb{N}$ s.t. $n \geq N_2 \Rightarrow |b_n - b| < \varepsilon/2$

Take $N = \max\{N_1, N_2\}$, if $n \geq N$ then

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(ii) Since $a_n \to a$, $\forall \varepsilon > 0$ $\exists N_1 \in \mathbb{N}$ s.t. $n \geq N_1 \Rightarrow |a_n - a| < \sqrt{\varepsilon}$

Since $b_n \to b$, $\forall \varepsilon > 0$ $\exists N_2 \in \mathbb{N}$ s.t. $n \geq N_2 \Rightarrow |b_n - b| < \sqrt{\varepsilon}$

Take $N = \max\{N_1, N_2\}$.

$$a_n \cdot b_n - a \cdot b = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a)$$

if $n \geq N$, then

$$|(a_n-a)(b_n-b)| = |a_n-a| \cdot |b_n-b| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon.$$

So $(a_n-a)(b_n-b) \longrightarrow 0$.

By (i) and (iii), we have
$$a(b_n-b) \longrightarrow 0$$
$$b(a_n-a) \longrightarrow 0$$

so $a_n \cdot b_n - a.b \longrightarrow 0+0+0=0$; i.e., $a_n \cdot b_n \longrightarrow a.b$.

(iii) Pf is an Exercise.

⚠ Cannot use (ii) to prove (iii)!

(iv) Since $a_n \longrightarrow a$, $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - a| < \varepsilon \frac{|a|^2}{2}$.

Taking $\varepsilon = \frac{1}{2}|a|$, $\exists N_1$ s.t. if $n \geq N_1$ we have $|a_n-a| < \frac{1}{2}|a|$.

Then $|a| = |a_n - (a-a_n)| \leq |a_n| + |a-a_n| < |a_n| + \frac{1}{2}|a|$

So, subtracting $\frac{1}{2}|a|$ from both sides, we have

$$\frac{1}{2}|a| < |a_n| \qquad \text{if} \qquad n \geq N_1.$$

Thus, if $n \geq \max\{N, N_1\}$, then:

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n \cdot a}\right| < \frac{2|a_n - a|}{|a|^2} \leq \frac{2}{|a|^2} \cdot \varepsilon \frac{|a|^2}{2} = \varepsilon.$$

$\square$

so $\frac{1}{a_n} \to \frac{1}{a}$.

Remark: Sequences in $\mathbb{R}^k$

(n is index of sequence)

A sequence $x_n = (\underbrace{x_{1,n}}_{j=1}, \ldots, \underbrace{x_{k,n}}_{j=k})$, $n \in \mathbb{N}$, converges to

$x_\infty = (\underbrace{x_{1,\infty}}_{j=1}, \ldots, \underbrace{x_{k,\infty}}_{j=k}) \in \mathbb{R}^k$ if and only if

(j is index of coordinate)

$$x_{j,n} \to x_{j,\infty}, \qquad \forall j = 1, \ldots, k.$$

In particular, if $\{x_n\}$, $\{y_n\}$ are seq. in $\mathbb{R}^k$, such that

$x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x+y$

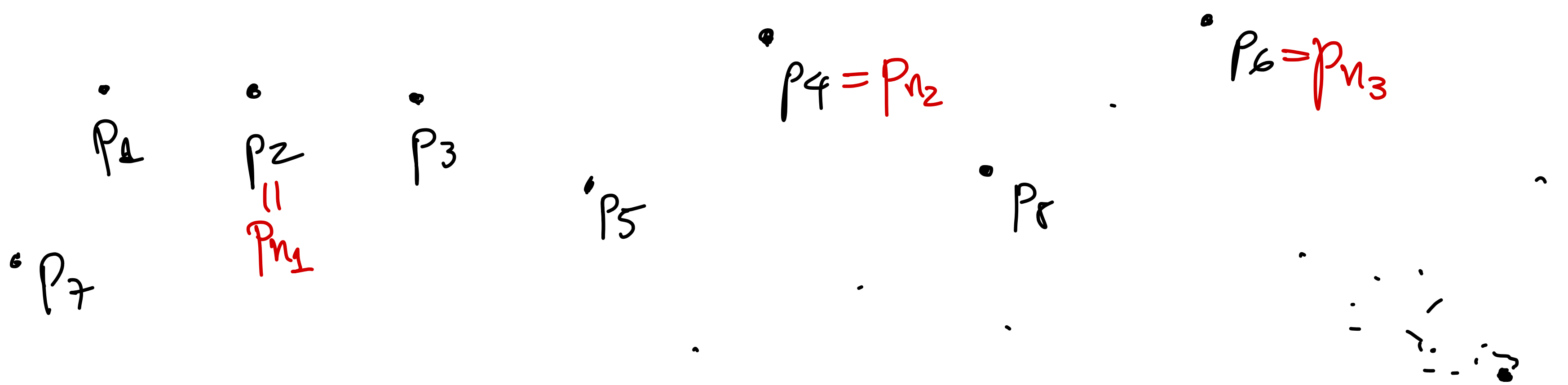
and $x_n \cdot y_n \to x \cdot y$

(dot product of vectors)

Subsequences

Def: Given a sequence $\{p_n\}$ in a metric space (X,d) a subsequence of $\{p_n\}$ is a sequence $\{p_{n_k}\}$, where $\{n_k\}$ is a sequence in $\mathbb{N}$.

Ex:



$n_1 = 2, \; n_2 = 4, \; n_3 = 6, \; \dots, \; n_k = 2k$

Fact: If $p_n \to p_\infty$, then any subseq. of p_n also converges to p_∞.

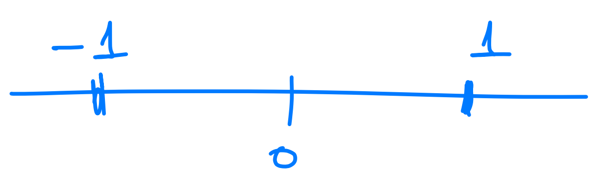
Thm: (a) If $\{p_n\}$ is a seq. in a <u>compact</u> metric space X, then there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ with $p_{n_k} \to p \in X$.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Pf: (a) Let $E = \{p_n : n \in \mathbb{N}\} \subset X$. If $|E| < \infty$, then there exists $p \in E$ and a seq. $n_1 < n_2 < \dots < n_k < \dots$ of natural numbers s.t. $p_{n_1} = p_{n_2} = \dots = p_{n_k} = \dots = p$

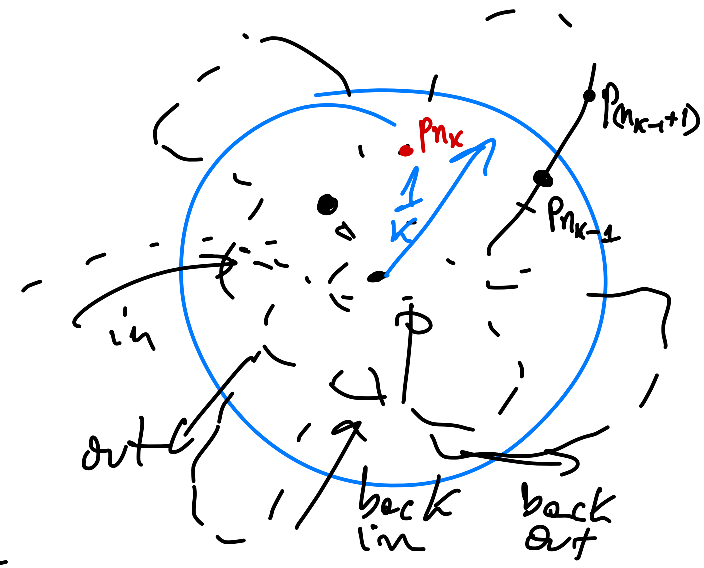
(Ex: $p_n = (-1)^n$

$E = \{+1, -1\}$



$p_n = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$)

and clearly $p_{n_k} \longrightarrow p$. If $|E| = \infty$, then by Video 8 of Lecture 5, it follows that E, being an infinite subset of the compact metric space X, has a limit point $p \in X$. Choose n_1 s.t. $d(p_{n_1}, p) < 1$. Next, suppose we chose $n_1, \ldots, n_{k-1}$ in such way that $d(p_{n_j}, p) < \frac{1}{j}$; for all $j = 1, \ldots, k-1$. Choose $n_k > n_{k-1}$ s.t. $d(p_{n_k}, p) < \frac{1}{k}$, because every neighbd. of $p \in X$ has infinitely many elements of E. This defines a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ s.t. $p_{n_k} \longrightarrow p$.



(b) Since every k-cell is compact; and every bounded subset $E \subset \mathbb{R}^k$ is (by definition) contained in a k-cell, a sequence $\{p_n\}$ in E is, in particular, a sequence in this k-cell, and hence admits a convergent subsequence by (a).

$\square$

Def: Limits of subsequences of $\{p_n\}$ are called subsequential limits of $\{p_n\}$.

Note: If $\{p_n\}$ converges, $p_n \to p_\infty$, then p_∞ is the only subsequential limit of $\{p_n\}$.
There also exist (divergent) sequences $\{p_n\}$ with distinct subsequential limits.

Ex: $a_n = (-1)^n$ does not converge, but 1 and -1 are both subsequential limits.

Thm: Given a seq. $\{p_n\}$ in a metric space (X, d), the set $E^* = \{p \in X : p \text{ is a subseq. limit of } \{p_n\}\}$ is closed.

Pf: Let $q \in X$ be a limit point of E^*. We need to show that $q \in E^*$.

if $|E^*| = 1$, then E^* is closed, so nothing to prove.

Choose n_1 so that $p_{n_1} \neq q$. Let $\delta = d(p_{n_1}, q) > 0$.

Suppose $n_1, \dots, n_{i-1}$ were chosen w/ $d(p_{n_j}, q) < \frac{\delta}{2^{j-1}}$ for all $j = 1, \dots, i-1$. Since q is a limit point of E^*, $\exists x \in E^*$ with $d(x, q) < \frac{\delta}{2^i}$. Since $x \in E^*$

there is $n_i > n_{i-1}$ s.t.

$$d(x, p_{n_i}) < \frac{\delta}{2^i}$$

Thus

$$d(q, p_{n_i}) \le d(q, x) + d(x, p_{n_i})$$
$$< \frac{\delta}{2^i} + \frac{\delta}{2^i} = \frac{\delta}{2^{i-1}}.$$

(so $d(p_{n_i}, q) \to 0$, i.e., $p_{n_i} \to q$.)

This gives a subseq $\{p_{n_i}\}$ s.t. $p_{n_i} \to q$; hence $q \in E^*$.

$\square$