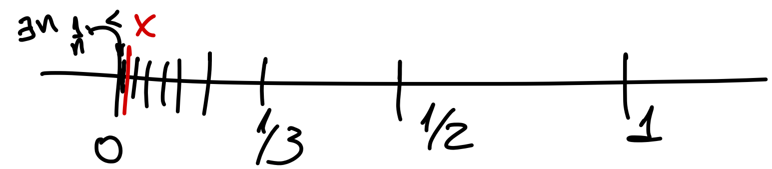
MAT320 Lecture 7 9/21/2020

Discussion of HW 1:

#1 $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots \right\}$

$\inf E = 0$



(Archimedean prop : $\forall x, y \in \mathbb{R},\ x > 0,\ \exists n \in \mathbb{N}\ \ nx > y$.)

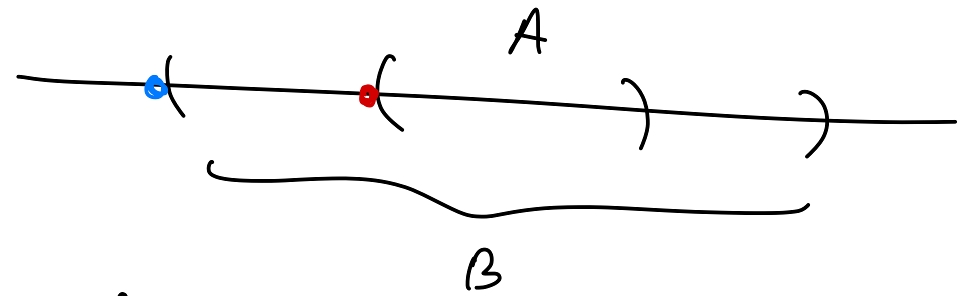
1. 0 is a lower bound for E: $\forall n \in \mathbb{N}\ \ \frac{1}{n} > 0$.

2. 0 is the largest lower bound for E:

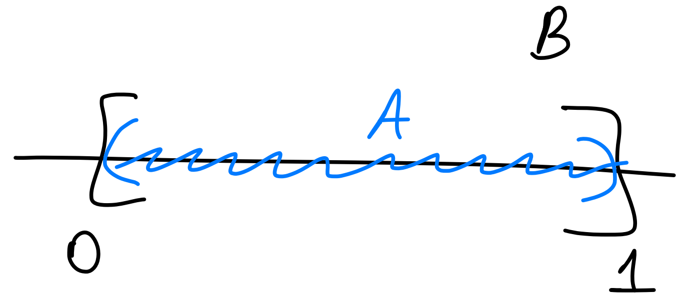
Suppose $x = \inf E$, $x > 0$. Then by Archimedean prop (w/ $y = 1$) $\exists n \in \mathbb{N}\ \ nx > 1$; that is, $x > \frac{1}{n}$.

This contradicts that x is a lower bound for E. $\square$

#2 $A \subset B \subset \mathbb{R}$ bounded $\Longrightarrow \inf B \le \inf A \le \sup A \le \sup B$



Examples: Find $A, B \subset \mathbb{R}$
as above with $A \neq B$ $(A \subsetneq B)$
and $\inf A = \inf B$.



For example: $B = [0, 1]$

$\inf B = 0.$

$A = (0, 1)$

$\inf A = 0.$

$0 \notin A$ but $0 \in B$.

#3 $f, g : X \subset \mathbb{R} \longrightarrow \mathbb{R}$

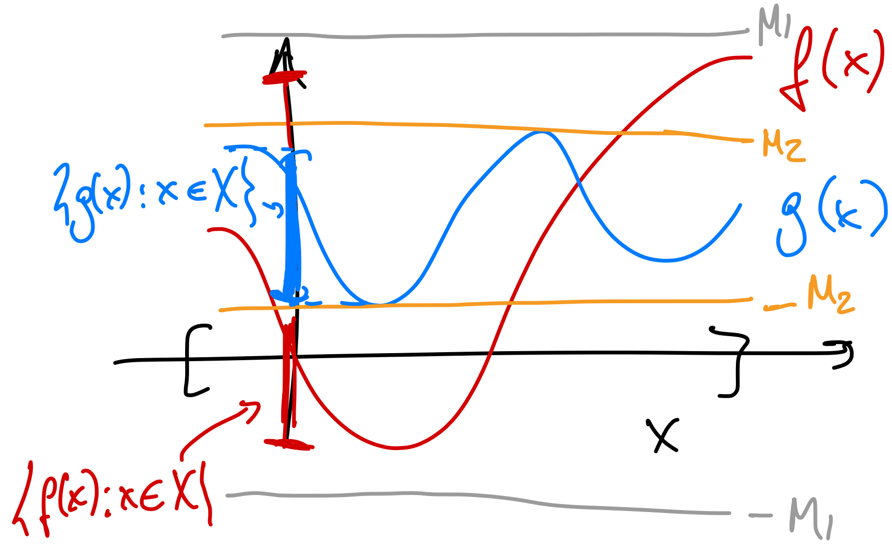
bounded:

$\exists M_1 \in \mathbb{R}$

$-M_1 \leq f(x) \leq M_1, \quad \forall x \in X$

$\exists M_2 \in \mathbb{R}$

$-M_2 \leq g(x) \leq M_2, \quad \forall x \in X.$



$\Longrightarrow f+g$ bounded b/c $(f+g)(x) = f(x) + g(x)$

$$-M_1 - M_2 \leq f(x) + g(x) \leq M_1 + M_2$$

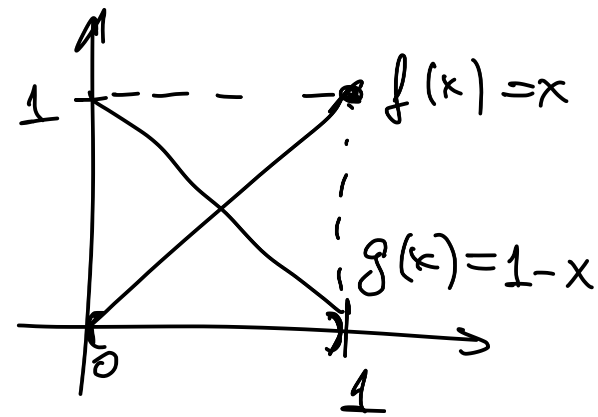
Let $M = M_1 + M_2$, then $\quad -M \leq f(x) + g(x) \leq M$

so $f+g$ is bounded. $\quad \left(\Longleftrightarrow |f(x) + g(x)| \leq M \right)$

Can argue $\sup(f+g) \leq \sup f + \sup g$ by showing that $\sup f + \sup g$ is an upper bound for $f+g$, and since $\sup(f+g)$ is the least upper bound, then $\sup(f+g) \leq \sup f + \sup g$.

#4 Example: $f(x) = x$, $g(x) = 1-x$, on $X = (0,1)$.

$$\sup f = \sup g = 1$$



$$(f+g)(x) = x + (1-x) = 1, \quad \forall x \in X$$

$$\sup(f+g) = 1 < \sup f + \sup g = 1 + 1 = 2$$

Simple exercise:

If $x \in \mathbb{R}$ is "smaller than any positive real number", i.e.

$$x \in \mathbb{R} \text{ is s.t. } \forall \varepsilon > 0, \quad x < \varepsilon$$

Then: $x \leq 0$.

Pf: Suppose, by contradiction, that $x \in \mathbb{R}$ is s.t. $\forall \varepsilon > 0$, $x < \varepsilon$ but $x > 0$. Then let $\varepsilon = x/2$; since this ε is positive and, if $x > 0$ and $\varepsilon = x/2$, then $\varepsilon < x$, this would contradict the above property $(\forall \varepsilon > 0,\ x < \varepsilon)$.

Therefore, we must have $x \leq 0$. $\square$

Note that: it is also true that if $x \in \mathbb{R}$, s.t.

$$x < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}, \quad \text{then} \quad x \leq 0.$$

Rudin Chap 2 Exercise #2

<u>Def</u>: A real number $x_0 \in \mathbb{R}$ is <u>algebraic</u> if there exists a polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$, with $a_j \in \mathbb{Q}$, $j = 0, \ldots, n$, and s.t. $p(x_0) = 0$.

<u>E.g.</u>: $\sqrt{2}$ is algebraic: $p(x) = x^2 - 2$. $p(\sqrt{2}) = 0$.

$\sqrt{3}$, $\sqrt{n}$, $n \in \mathbb{N}$

π is not algebraic, e is not algebraic.

<u>Exercise</u>: Show that $A = \{x \in \mathbb{R} : x \text{ is algebraic}\}$ is countable.

<u>Sol</u>:

$$A = \bigcup_{p(x) \in \mathbb{Q}[x]} \{x \in \mathbb{R} : p(x) = 0\}$$

Note that this is a <u>finite</u> set for each given $p(x) \in \mathbb{Q}[x]$.
$\#\{x \in \mathbb{R} : p(x) = 0\} \leq \text{degree}(p)$.

$\mathbb{Q}[x] = \{$polynomials w/ rational coeff., in the variable $x\}$.

So it suffices to show that $\mathbb{Q}[x]$ is countable, for then A is a countable union of finite sets (and hence countable) $\longleftarrow$ Lecture 3, Video 5

Def. $\mathbb{Q}[x]_n = \{ \underbrace{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}_{p(x)} : a_j \in \mathbb{Q} \}$ $\leftarrow$ polynomials of fixed degree n.

$$\mathbb{Q}[x]_n \xrightarrow{\cong} \mathbb{Q}^{n+1} = \underbrace{\mathbb{Q} \times \cdots \times \mathbb{Q}}_{(n+1)\text{-times}} \qquad p(x) \longmapsto (a_n, a_{n-1}, \ldots, a_1, a_0) \in \mathbb{Q}^{n+1}$$

Are (finite) cartesian products of countable sets ($\mathbb{Q}$ is countable)

$\Downarrow$

$\mathbb{Q}^{n+1}$ is countable

a bijection, so $\mathbb{Q}[x] = \bigcup_{n \in \mathbb{N}} \mathbb{Q}[x]_n = \bigcup_{n \in \mathbb{N}} \mathbb{Q}^{n+1}$

So $\mathbb{Q}[x]$ is a union of countably many countable sets; hence countable. $\square$

Cor: The set $\mathbb{R} \setminus \mathbb{A}$ of transcendental numbers is uncountable.

Pf: $\mathbb{R} = (\mathbb{R} \setminus \mathbb{A}) \cup \mathbb{A}$

uncountable (Lecture 6) ... countable (above)

So $\mathbb{R} \setminus \mathbb{A}$ cannot be countable; if it was, would lead to a contradiction...

Exercise: Let (X,d) be a metric space.

Useful for HW 2 #3

1) Is d^2 a distance function on X? No

2) Is $5d$ a distance function on X? YES.

1) Let $D(x,y) = d(x,y)^2$, $\forall x,y \in X$ (i.e. $D = d^2$)

- $D(x,y) \geq 0$, $\forall x,y \in X$ $D(x,y) = 0 \iff x = y$ ✓

$D(x,y) = d(x,y)^2 \geq 0$ $0 = D(x,y) = d(x,y)^2 \iff d(x,y) = 0$

$\iff x = y$

d is a dist. funct.

- $D(x,y) = D(y,x)$, $\forall x,y \in X$

$D(x,y) = (d(x,y))^2 = (d(y,x))^2 = D(y,x)$ ✓

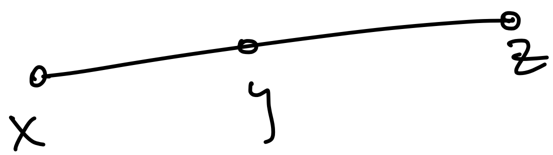
d is a dist. fct.

- Triangle inequality: $D(x,z) \leq D(x,y) + D(y,z)$

$d(x,z) \leq d(x,y) + d(y,z)$ (b/c d is a dist fct.)

$\Downarrow$

$d(x,z)^2 \leq (d(x,y) + d(y,z))^2 = d(x,y)^2 + d(y,z)^2 + \underline{2\, d(x,y)\, d(y,z)}$



Suppose equality holds and $x \neq y \neq z$

$$d(x,z) = d(x,y) + d(y,z)$$

Then:

$$D(x,z) = d(x,z)^2 = d(x,y)^2 + d(y,z)^2 + 2\,d(x,y)\,d(y,z)$$

$$= D(x,y) + D(y,z) + \underbrace{2\,d(x,y)\,d(y,z)}_{>0}$$

This would lead to the following contradiction:

$$D(x,y) + D(y,z) + \underbrace{2\,d(x,y)\,d(y,z)}_{>0} = D(x,z) \overset{\text{Triangle ineq.}}{\leq} D(x,y) + D(y,z)$$

$$\Downarrow$$

$$0 < 2\,d(x,y)\,d(y,z) \leq 0.$$

$\square$

2) $E(x,y) := 5d(x,y), \quad \forall x,y \in X.$

- $E(x,y) \geq 0 \quad \forall x,y \in X$, and $E(x,y) = 0 \iff x = y$

$E(x,y) = 5d(x,y) \geq 0$ d is a dist ✓

$0 = E(x,y) = 5d(x,y) \iff d(x,y) = 0 \iff x = y$

- $E(x,y) = E(y,x), \quad \forall x,y \in X$

$E(x,y) = 5d(x,y) = 5d(y,x) = E(y,x)$

- Triangle inequality: ✓

$E(x,z) \leq E(x,y) + E(y,z), \quad \forall x,y,z \in X$

$$\begin{array}{ccccc} E(x,z) & \leq & E(x,y) & + & E(y,z) \\ \| & & \| & & \| \\ 5d(x,z) & & 5d(x,y) & & 5d(y,z) \end{array}$$

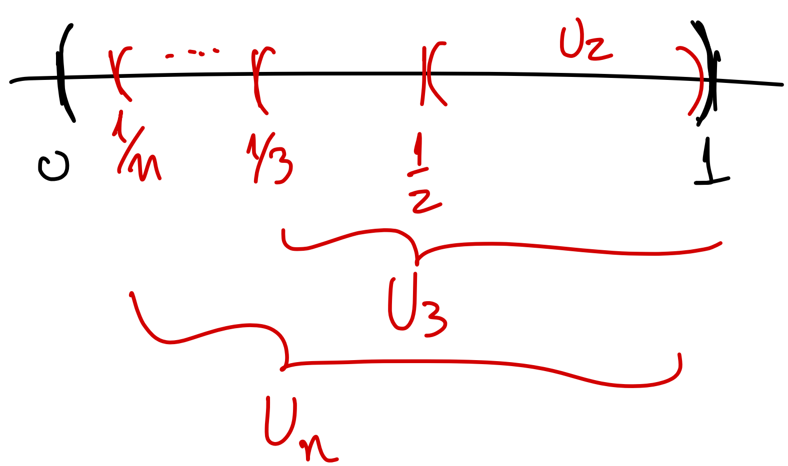
B/c d is a dist. funct: $5d(x,z) \leq 5\big(d(x,y) + d(y,z)\big) = \begin{array}{c} 5d(x,y) \\ + \\ 5d(y,z) \end{array}$

Rudin Chap 2 Exercise # 14

Find an open cover of $E=(0,1)$ without finite subcover.

Recall: $E=(0,1)$ is not compact, so such an open cover exists.

Sol: Let $U_n = \left(\frac{1}{n}, 1\right)$ for all $n \in \mathbb{N}$



$U_1 = \emptyset$

$U_2 = \left(\frac{1}{2}, 1\right)$

$U_3 = \left(\frac{1}{3}, 1\right)$

$\vdots$

$U_n = \left(\frac{1}{n}, 1\right)$

$(0,1) \subset \bigcup_{n \in \mathbb{N}} U_n$, i.e., $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of $(0,1)$.

- Clearly U_n is open for all $n \in \mathbb{N}$

- $\forall x \in (0,1)$ $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x < 1$, i.e., $x \in U_n$, by the Archimedean property.

Any finite subset of $\{U_n\}_{n\in\mathbb{N}}$ <u>does</u> <u>not</u> cover $E = (0,1)$: let $\{U_{n_1}, U_{n_2}, \dots, U_{n_k}\}$ be such a finite subcollection of open subsets. Let

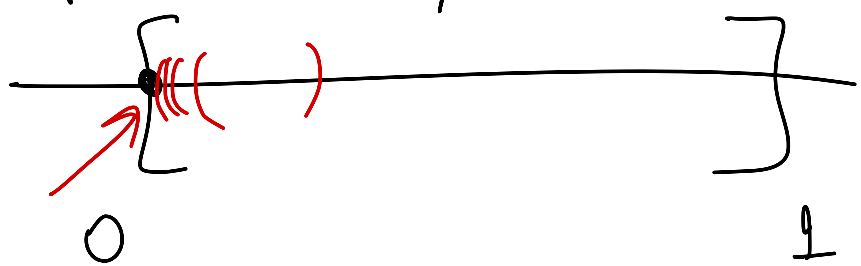
$$n = \max\{n_1, \dots, n_k\} \in \mathbb{N}$$

Then $\frac{1}{n+1} \notin \bigcup_{j=1}^{k} U_{n_j} = U_{n_1} \cup \dots \cup U_{n_k}$ because

$$\frac{1}{n+1} < \frac{1}{n_j} \quad \forall j = 1, \dots, k \quad \text{but} \quad \frac{1}{n+1} \in E.$$

Note: If we were trying to do this on the compact set $\overline{E} = [0,1]$, then it wouldn't work:

$\rightarrow$ $0 \in \overline{E}$ would have to be in some open set of the open cover



and, indeed, $[0,1]$ is compact, so every open cover has a finite subcover.