MAT320 Lecture 6 9/16/2020

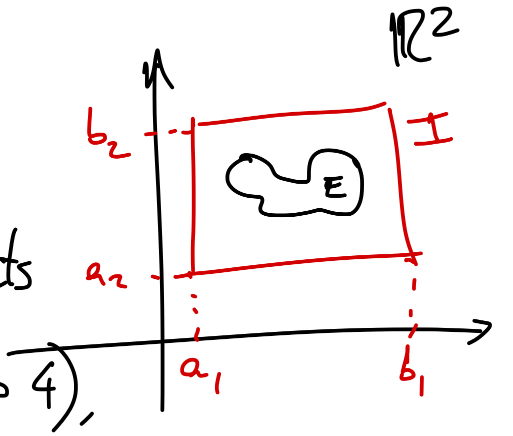
Heine–Borel Theorem: The following are equivalent for a subset $E \subset \mathbb{R}^k$ of Euclidean space:

(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset $S \subset E$ has a limit point in E.

Pf: (a) $\Rightarrow$ (b) If E is closed and bounded, then these exists a k-cell $I \subset \mathbb{R}^k$ such that $E \subset I$.



From last lecture: I is compact.

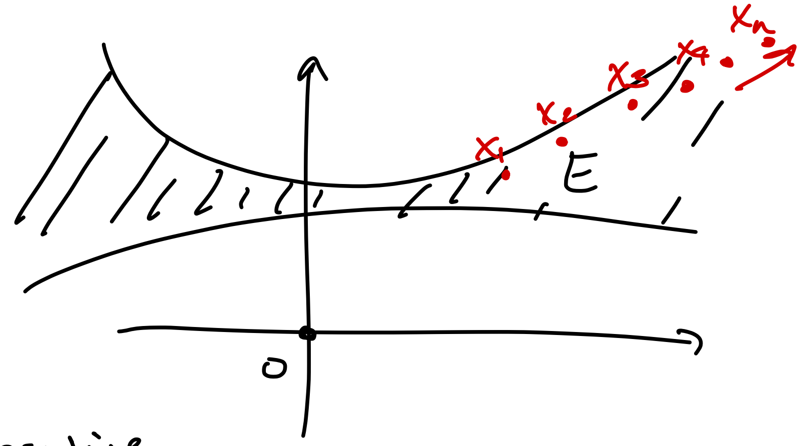
Since E is closed in I, and closed subsets of a compact set are compact (Lecture 5, video 4), it follows that E is compact.

(b) $\Rightarrow$ (c) Video 8 of Lecture 5.

(c) $\Rightarrow$ (a) If E satisfies (c) but is not bounded, then

$\exists x_n \in E,\ \forall n \in \mathbb{N}$ s.t.

$\|x_n\| > n$

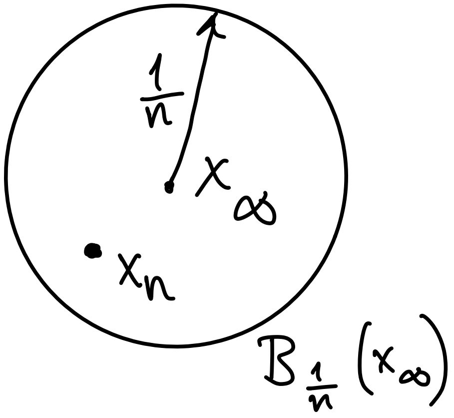


Let $S = \{x_n \in E : n \in \mathbb{N}\}$.

Note that S is infinite since, otherwise, $\{\|x_n\| : n \in \mathbb{N}\}$ would be finite (which it is not). By (c), there is a limit point of S in E. But S does not have any limit points in $\mathbb{R}^k$ (nor in E), which gives the desired contradiction.

Suppose that E is not closed, then $\exists x_\infty \in \mathbb{R}^k$ a limit point of E with $x_\infty \notin E$. Let $x_n \in E$ be s.t.

$\|x_n - x_\infty\| < \frac{1}{n}$. Let $S = \{x_n \in E : n \in \mathbb{N}\}$.



Note that S is infinite (otherwise $\|x_n - x_\infty\|$ would be a positive constant) //

clearly x_∞ is a limit point of S, and S has no other limit point: if $y \in \mathbb{R}^k$, $y \neq x_\infty$, was a limit point of S, then

$$\|x_n - y\| \overset{\text{(Triangle ineq.)}}{\geq} \|x_\infty - y\| - \|x_n - x_\infty\|$$

$$> \|x_\infty - y\| - \frac{1}{n} \underset{\text{for } n \in \mathbb{N} \text{ suff. large.}}{\geq} \frac{1}{2}\|x_\infty - y\|$$

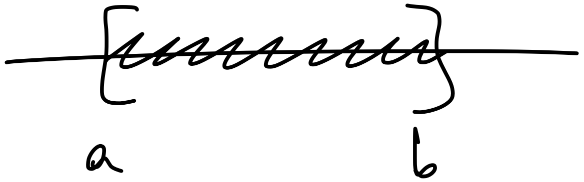
contradicting the assumption that y is a limit point of S.

This contradicts (c), b/c $S \subset E$ is an infinite subset without a limit point in E. Therefore, E must also be <u>closed</u>.

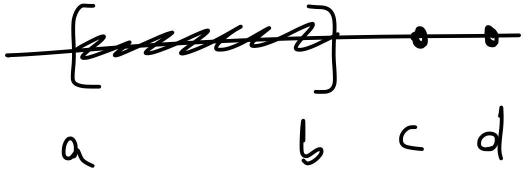
$\square$

Recall: $P \subset (X,d)$ is a *perfect* set if P is closed and every point of P is a limit point of P.

Ex: $P = [a,b] \subset \mathbb{R}$ is perfect



$E = [a,b] \cup \{c,d\}$ is not perfect



closed ✓

$c \in E$ is not a limit point of E.

Theorem: If $P \subset \mathbb{R}^k$ is perfect, $P \neq \emptyset$, then P is *uncountable*.

Pf: Since P has limit points, it is infinite. Suppose that P is countable, and label its elements as $x_1, x_2, \ldots$, i.e.,

$$P = \{x_1, x_2, x_3, \ldots\} = \{x_n : n \in \mathbb{N}\}.$$

Construct a sequence of neighborhoods $\{V_n\}$ as follows:

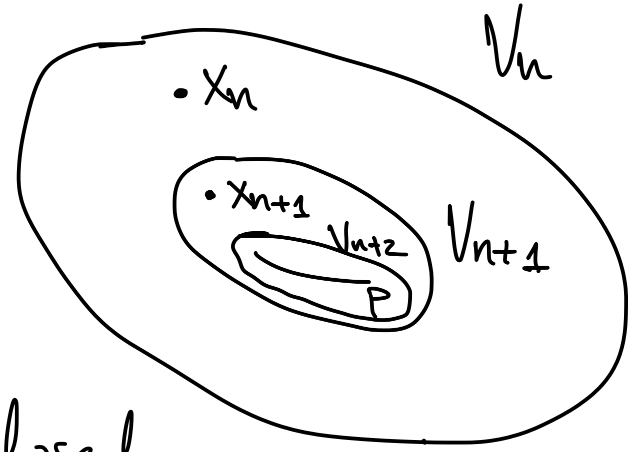
$V_1 \ni x_1$ is any neighbd. of x_1.

By induction, suppose we constructed $V_1, \dots, V_n$ in such way that $V_n \cap P \neq \emptyset$. Since $\forall p \in P$, p is a limit point of P, there exists a neighborhood V_{n+1} s.t.

(i) $\overline{V_{n+1}} \subset V_n$

(ii) $x_n \notin \overline{V_{n+1}}$

(iii) $V_{n+1} \cap P \neq \emptyset$.

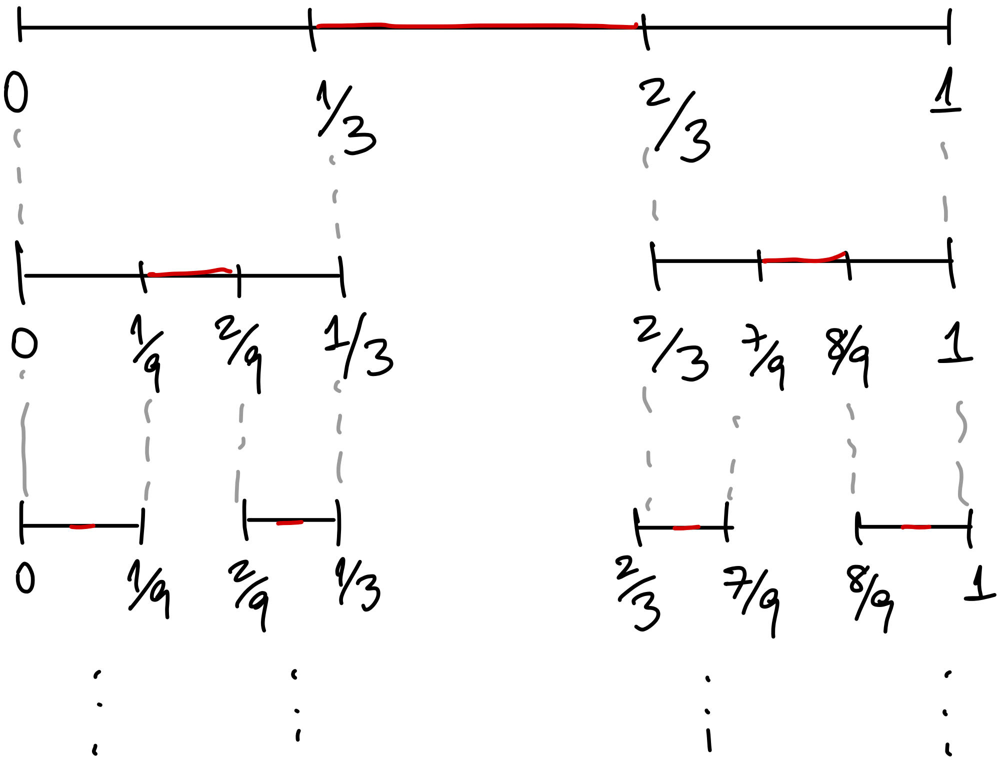


Let $K_n = \overline{V_n} \cap P$. Since $\overline{V_n}$ is closed and bounded, by the Heine-Borel Theorem, $\overline{V_n}$ is compact. Since $x_n \notin K_{n+1}$, no point of P lies $\bigcap_{n \in \mathbb{N}} K_n$. Since $K_n \subset P$, $\bigcap_{n \in \mathbb{N}} K_n = \emptyset$. But each K_n is nonempty, and $K_n \supset K_{n+1}$. This contradicts the fact that intersections of nested seq. of compact sets are nonempty (Video 6 of Lecture 5). □

Cor: Any closed interval $[a,b] \subset \mathbb{R}$ is uncountable.

Cor: The set of real numbers $\mathbb{R}$ is uncountable.

The Cantor Set



$E_0 = [0,1] \leftarrow$ 1 closed interval of length 1

$E_1 = [0,1/3] \cup [2/3,1] \leftarrow$ 2 closed intervals of length 1/3

$$\left.\begin{aligned} E_2 = {} & [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup \\ & \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1] \end{aligned}\right\} \leftarrow 2^2 = 4 \text{ closed intervals of length } 1/3^2$$

Continue by induction, constructing a nested sequence E_n s.t.:

(i) $E_0 \supset E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$

(ii) E_n is the union of 2^n closed intervals, each of length $\frac{1}{3^n}$.

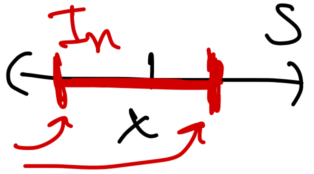
Def: The *Cantor set* is $P = \bigcap_{n \in \mathbb{N}} E_n$.

Note that P is the intersection of nested nonempty compact sets, hence $\underline{P \neq \phi}$. (Video 6 of Lecture 5).

Clearly, P is bounded and closed, hence P is *compact*.

Prop: The Cantor Set P is perfect.

Pf: Clearly P is closed, so it is enough to show that any $x \in P$ is a limit point of P. Let S be any neighbhd of $x \in P$. By choosing $n \in \mathbb{N}$ sufficiently large, so that one of the intervals I_n that



endpoints belong to P.

constitute E_n satisfies $I_n \subset S$, we have that its endpoints (which are elements of P) belong to S. $\square$

<u>Remark</u>: Note that P does not contain any interval (α, β): no interval of the form

$$I_{k,m} := \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right)$$

← These are the "middle thirds" that were removed in step m of the construction

has any point in common with P, for any $k, m \in \mathbb{N}$; but $I_{k,m}$ is inside (α, β) for some $k \in \mathbb{N}$ if m is chosen sufficiently large: $\frac{1}{3^m} < \frac{\beta - \alpha}{6}$.

<u>Exercise</u>: Compute the total length of intervals that are removed from $E_0 = [0,1]$ in the construction of the Cantor set P.

In step n of the construction, we remove 2^n intervals of length $1/3^{n+1}$:

$n=0$: remove 1 interval of length $1/3$

$n=1$: —"— 2 —"— $1/3^2$

$n=2$: —"— 2^2 —"— $1/3^3$

$\vdots$

n —"— 2^n —"— $1/3^{n+1}$

$\vdots$ $\vdots$ $\vdots$

$$\left(\begin{array}{l}\text{Total length}\\ \text{removed}\end{array}\right) = \sum_{n=0}^{+\infty} \underbrace{2^n}_{\substack{\uparrow\\ \text{\# of intervals}\\ \text{removed}}} \cdot \underbrace{\frac{1}{3^{n+1}}}_{\substack{\uparrow\\ \text{length of}\\ \text{each one}}} = \frac{1}{3}\sum_{n=0}^{+\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-2/3}\right)$$

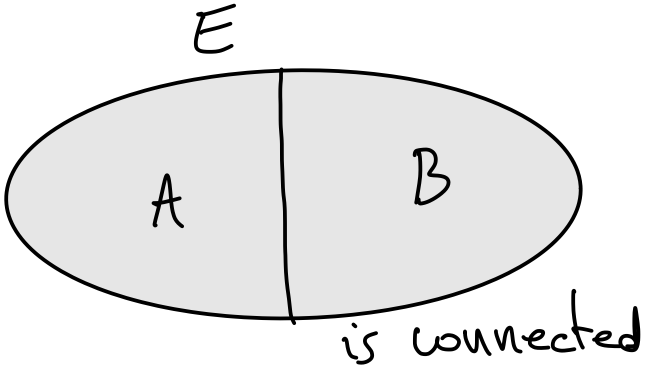
$$= \frac{1}{3}\cdot\frac{1}{1/3}$$

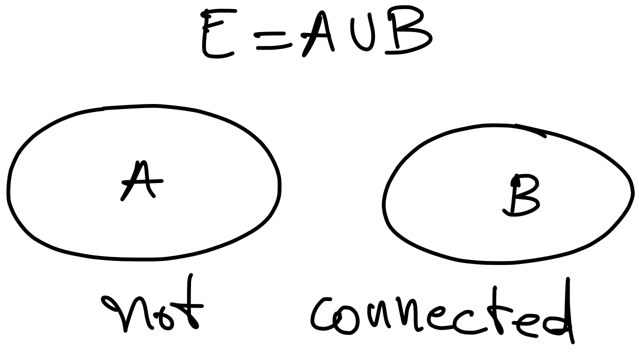
$$= \boxed{1}$$

So, remarkably, we began with $E_0=[0,1]$, which has length 1, removed intervals whose total length is also 1, and the resulting set P is perfect (in particular, P is uncountable).

Connected sets:

Def: Two subsets A and B of a metric space X are said to be separated if both $\overline{A}\cap B$ and $A\cap\overline{B}$ are empty. A set $E\subset X$ is connected if E cannot be written as $E=A\cup B$ where A and B are separated and nonempty.





Rmk: Separated $\Rightarrow$ disjoint
$\not\Leftarrow$

$A = [0,1]$ $\quad A \cap B = \phi$

$B = (1,2)$ $\quad$ (disjoint)

$\overline{B} = [1,2]$ $\quad \phi \neq A \cap \overline{B} = \{1\}$

(not separated!)

Thm: A subset $E \subset \mathbb{R}$ is connected if and only if E is an interval: "$\forall x, y \in E$, if z satisfies $x < z < y$, then $z \in E$."

That means $E = [a,b]$ or (a,b) or $[a,b)$ or $(a,b]$.

(Note: The negation of the above property reads: $\exists x, y \in E$ s.t. $x < z < y$ but $z \notin E$.)

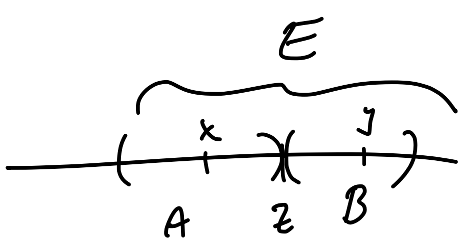
Pf: If E is not an interval, then $\exists x, y \in E$, $x < z < y$, $z \notin E$. Then we can write $E = A_z \cup B_z$,

where
$$A_z = E \cap (-\infty, z)$$
$$B_z = E \cap (z, +\infty)$$

Note that $x \in A_z$, $y \in B_z$ so $A_z \neq \emptyset$, $B_z \neq \emptyset$; moreover $A_z \subset (-\infty, z)$ and $B_z \subset (z, +\infty)$, so $\overline{A_z} \cap B_z = \emptyset$ and $A_z \cap \overline{B_z} = \emptyset$. Thus A_z and B_z are separated, hence E is not connected.

Conversely, suppose E is not connected: then $E = A \cup B$, with $A \neq \emptyset$, $B \neq \emptyset$ separated. Pick $x \in A$, $y \in B$, w/o loss of generality, say $x < y$. Define

$$z = \sup \left(A \cap [x, y] \right)$$



Note that $z \in \overline{A}$, hence $z \notin B$. In particular $x \leq z < y$.

If $z \notin A$, then $x < z < y$ and $z \notin E$. (so E is not an interval.) If $z \in A$, then $z \notin \overline{B}$; so there exists z_1 with $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$, and $z_1 \notin E$. (so, again, E is not an interval).

□