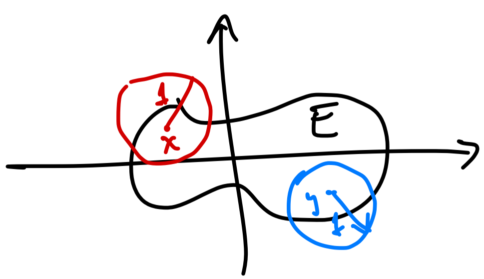
Compact Sets

Let (X,d) be a metric space.

Def: An open cover of $E \subset X$ is a collection G_α of open sets (in X) such that $E \subset \bigcup_\alpha G_\alpha$.

Ex: $(X,d) = (\mathbb{R}^2, d)$



$$\bigcup_{p \in E} B_1(p) \supset E$$

Def: A subset $K \subset X$ is compact if given any open cover $\{G_\alpha\}$ of K, there exists a finite subcover, say $G_{\alpha_1}, \dots, G_{\alpha_n}$ s.t. $K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n} =: \bigcup_{i=1}^{n} G_{\alpha_i}$.

Compactness relative to ...?

[**Recall:** A subset $E \subset Y \subset X$ is <u>open in Y</u> if and only if $E = Y \cap G$ where G is <u>open in X</u>.]

Similarly for closed subsets relative to a subspace.

Thm: If $K \subset Y \subset X$, then K is compact "in X" if and only if K is compact "in Y."

Pf: Suppose K is compact in X, let $\{V_\alpha\}$ be a collection of open subsets of Y s.t. $K \subset \bigcup_\alpha V_\alpha$. By a theorem in previous lecture (see remark above), there exist $\{G_\alpha\}$ open subsets of X s.t. $V_\alpha = G_\alpha \cap Y$. Since K is compact in X, we have that $K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$, for some $\alpha_1, \dots, \alpha_n$.

Since $K \subset Y$, we have that $K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$. Thus K is compact in Y.

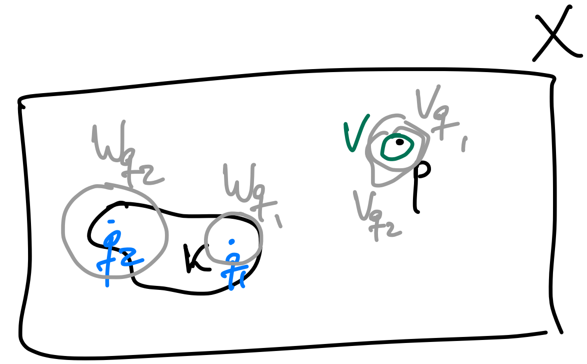
Conversely, suppose K is compact in Y. So let $\{G_\alpha\}$ be an open cover of K in X. Let $V_\alpha := G_\alpha \cap Y$. Since $K \subset Y$, $\{V_\alpha\}$ is an open cover of K in Y. Thus, since K is compact in Y, $K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ for some $\alpha_1, \dots, \alpha_n$. Then

$$K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} = \left(G_{\alpha_1} \cup \dots \cup G_{\alpha_n}\right) \cap Y \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

Therefore K is compact in X. $\square$

<u>Remark</u>: By the Theorem above, it makes sense to talk about "compact metric spaces", since compactness is an <u>intrinsic</u> property, unlike being "open" or "closed".

Thm: Compact subsets of metric spaces are closed.

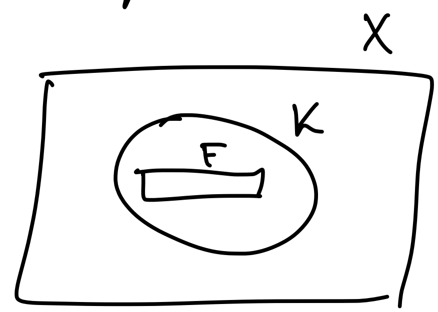


Pf: Let $K \subset X$ be compact, we would like to show that $K^c = X \setminus K$ is open.

Let $p \in K^c$, if $q \in K$, let $V_q \ni p$ and $W_q \ni q$ be open neighborhoods s.t. $V_q \cap W_q = \emptyset$. Note $\{W_q\}_{q \in K}$ is an open cover of K. Since K is compact, there are finitely many $q \in K$ s.t.

$K \subset W_{q_1} \cup \dots \cup W_{q_n} =: W$. Let $V := V_{q_1} \cap \dots \cap V_{q_n}$. Since this is an intersection of finitely many open subsets, V is open in X. Since $V_q \cap W_q = \emptyset$, $\forall q \in K$, we have $V \subset K^c$. Thus $p \in K^c$ is an interior point of K^c. Therefore K^c is open, i.e., K is closed. $\square$

Thm: Closed subsets of compact sets are compact.



Pf: Let $F \subset K \subset X$, F closed in X, K compact. Let $\{V_\alpha\}$ be an open cover of F. Note that

$$K \subset \left(\bigcup_\alpha V_\alpha\right) \cup F^c$$

open in X (b/c F is closed in X)

Since K is compact, there exists a finite subcover:

$$K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup F^c.$$

may or may not appear in the subcover.

Removing F^c if necessary, we have $F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$.

Thus, F is compact. $\square$

Cor: If F is closed and K is compact, then $F \cap K$ is compact.

Pf: $F \cap K$ is closed (since F and K are closed), and $F \cap K \subset K$ hence compact by the theorem above.

Thm: If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non-empty, then $\bigcap_\alpha K_\alpha$ is non-empty.

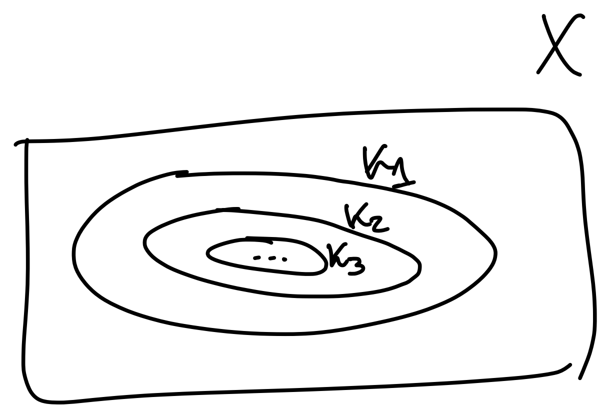
Pf: Fix K_1 a member of $\{K_\alpha\}$; set $G_\alpha = K_\alpha^c$. Suppose that no point in K_1 belongs to every K_α; i.e., $\forall p \in K_1$, $\exists \alpha$ s.t. $p \in K_\alpha^c = G_\alpha$. Since K_α are compact, they are closed, and hence G_α are open. So $\{G_\alpha\}$ is an open cover of K_1. Since K_1 is compact, there is a finite subcover:

$$K_1 \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} = K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_n}^c = (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n})^c$$

Thus: $K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \phi$, which contradicts the hypothesis. Therefore $\exists p \in K_1$ which belongs to all other K_α, i.e., $\{p\} \subset \bigcap_\alpha K_\alpha$.

$\square$

Cor: If $\{K_n\}$ is a <u>nested</u> sequence of nonempty compact subsets, i.e., $K_n \supset K_{n+1}$, $\forall n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$.



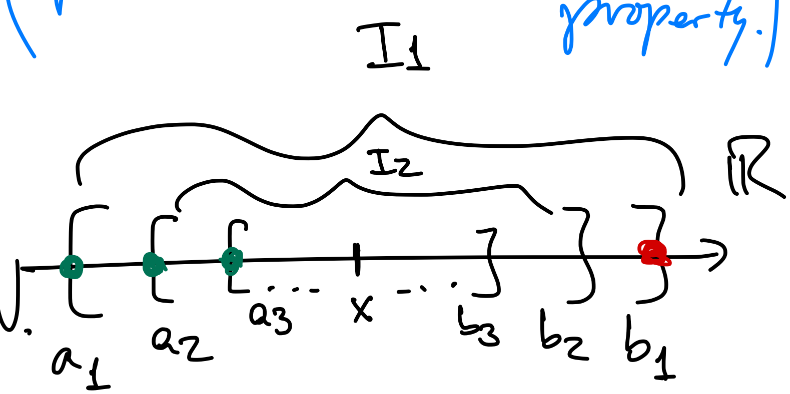
Pf: We just need to verify that the intersection of finitely many $\{K_n\}$ is nonempty:

$$K_{n_1} \cap K_{n_2} \cap \dots \cap K_{n_m} = K_{\max\{n_1, n_2, \dots, n_m\}} \neq \emptyset. \qquad \square$$

Example: Let $\{I_n\}$ be a nested sequence of closed intervals in $\mathbb{R}$, say $I_n = [a_n, b_n]$, $a_n < b_n$, $\forall n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$.

Pf: Let $E = \{a_n : n \in \mathbb{N}\} \neq \emptyset$. Clearly E is bounded from above, e.g., $b_1 \geq a_n$, $\forall n \in \mathbb{N}$, is an upper bound.

(Rmk: This is equiv. to the fact that $\mathbb{R}$ has the l.u.b. property.)



Let $x := \sup E$. Clearly, $x \geq a_n$, $\forall n \in \mathbb{N}$.

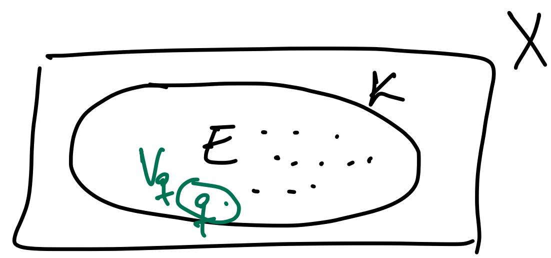
Moreover, for all $m, n \in \mathbb{N}$

$$a_n \leq a_{n+m} \leq b_{n+m} \leq \min\{b_n, b_m\} \leq b_m$$

i.e., $\forall m \in \mathbb{N}$, b_m is an upper bound for E. Since x is the <u>least</u> upper bound of E, $x \leq b_m$, $\forall m \in \mathbb{N}$. Therefore $a_n \leq x \leq b_n$, $\forall n \in \mathbb{N}$, i.e., $x \in \bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. $\square$

<u>Note</u>: The above proof <u>does not</u> use the fact (proven later) that I_n are compact!

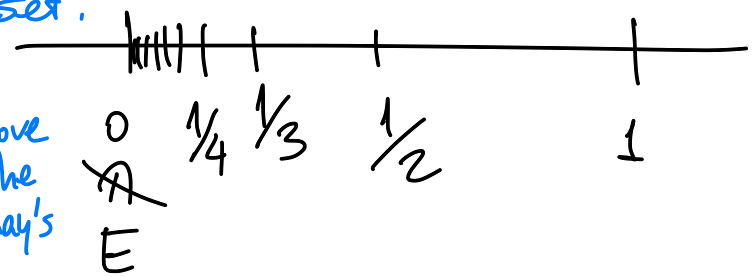
<u>Thm</u>: If E is an <u>infinite</u> subset of a compact set K, then E has a <u>limit point</u> in K.



<u>Pf</u>: Suppose no point of K is a limit point of E. Then $\forall q \in K$, $\exists V_q \ni q$ open neighbd. such that V_q contains at most one point of E (namely q itself). Note that, since E is infinite, no finite subcollection of $\{V_q\}$ can cover E, and, in particular, no finite

subcover of $\{V_\alpha\}$ can cover $K \supset E$. This contradicts the assumption that K is compact. $\square$

Example: $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ ← infinite set.

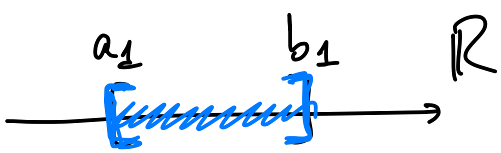


$K = [0,1]$ compact. ← We will prove this in the end of today's lecture!

Note that $0 \in K$ is a limit point of E in K.

k-cells in $\mathbb{R}^k$:

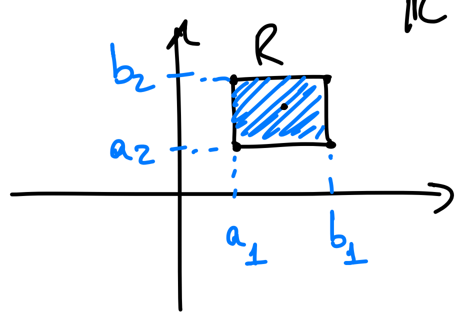
$k=1$



closed intervals

$k=2$

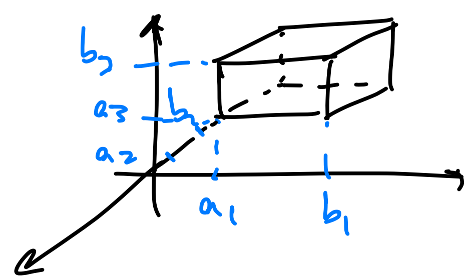
$\mathbb{R}^2$



$[a_1, b_1] \times [a_2, b_2]$

$R = \{ (x_1, x_2) \in \mathbb{R}^2 : a_i \le x_i \le b_i,\ i = 1,2 \}$

$k=3$



$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

$R = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : a_i \le x_i \le b_i,\ i = 1,2,3 \}$

<u>Def</u>: A k-cell in $\mathbb{R}^k$ is a subset of the form

$$[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k] = \left\{ (x_1, \dots, x_k) \in \mathbb{R}^k : \begin{array}{l} a_i \le x_i \le b_i \\ \forall i = 1, 2, \dots, k \end{array} \right\}.$$

Observe that, just like 1-cells (closed intervals), intersections of nested sequences of nonempty k-cells are nonempty.

<u>Pf</u>: Let $I_n = [a_{n,1}, b_{n,1}] \times [a_{n,2}, b_{n,2}] \times \dots \times [a_{n,k}, b_{n,k}]$ be a nested seq. of k-cells. Note that the <u>projections</u> of I_n to each coordinate axis x_j are collections of nested 1-cells, namely $[a_{n,j}, b_{n,j}]$. By previous example, we know $\bigcap_{n \in \mathbb{N}} [a_{n,j}, b_{n,j}] \neq \emptyset$, say $x_j \in \bigcap_{n \in \mathbb{N}} [a_{n,j}, b_{n,j}]$, $j = 1, \dots, k$.

Then $\vec{x} = (x_1, \dots, x_j, \dots, x_k) \in \bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

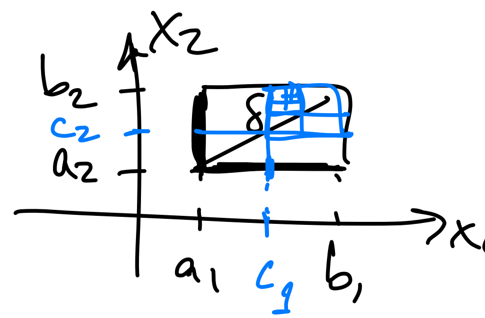
$\square$

Thm. k-cells are compact

(In particular, closed intervals $[a,b] \subset \mathbb{R}$ are compact).

Pf: Let $I = [a_1, b_1] \times \dots \times [a_k, b_k]$ be a k-cell. Let

$$\delta = \sqrt{\sum_{i=1}^{k} (b_i - a_i)^2}.$$ Note that



$\forall x, y \in I, \quad d(x,y) \leq \delta.$

Suppose, by contradiction, that I is not compact. Then there exists an open cover $\{G_\alpha\}$ of I which admits no finite subcover. Let $c_j = \dfrac{a_j + b_j}{2}$.

The k-cells determined by the intervals $[a_j, c_j]$ and $[c_j, b_j]$ subdivide I into 2^k

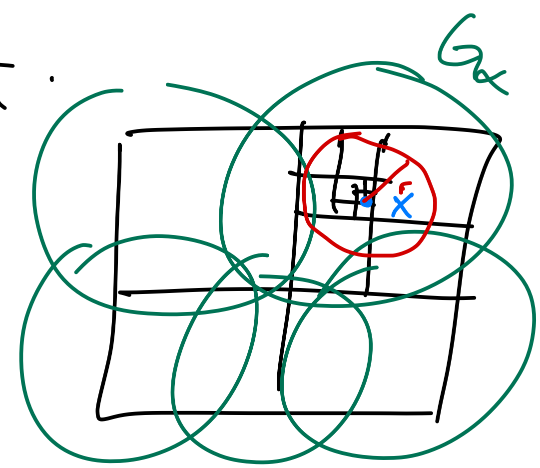
k-cells, say Q_i, whose union is I. At least one of Q_i cannot be compact, say Q_1 is not compact. Continue as before and subdivide $I_1 = Q_1$ into 2^k sub-k-cells determined by midpoints.

We obtain a sequence I_n, with $I_0 = I$, $I_1 = Q_1$, s.t.

(a) $I \supset I_1 \supset I_2 \supset \dots \supset I_n \supset \dots$

(b) I_n is not covered by a subcover of $\{G_\alpha\}$.

(c) If $x, y \in I_n$, then $d(x,y) \leq \frac{\delta}{2^n}$.



By (a), $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$, say $x \in \bigcap_{n\in\mathbb{N}} I_n$

For some α, $x \in G_\alpha$. Since G_α is open

$\exists r > 0$ s.t. $B_r(x) \subset G_\alpha$. In other words, $\forall y$ s.t. $d(x,y) < r$, $y \in G_\alpha$. So choosing n large enough, we have $\frac{\delta}{2^n} < r$ (by the Archimedean property of $\mathbb{R}$), then $I_n \subset G_\alpha$, by (c). This contradicts (b), and finishes the proof that I is compact. $\square$