MAT320 Lecture 26 12/2/2020

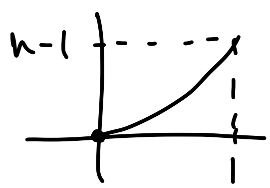
Weierstrass Approximation Theorem: If $f:[a,b]\to\mathbb{R}$ is a continuous function, then there exists a sequence $\{P_n\}$ of polynomials s.t. $P_n \to f$ uniformly on $[a,b]$.

(In other words, polynomials are dense in $\mathcal{C}([a,b],\mathbb{R})$.)

Lemma: $\forall x\in[0,1]$, $\forall n\in\mathbb{N}$, $(1-x^2)^n \geq 1-nx^2$.

Pf: Let $\phi(x) = (1-x^2)^n - 1 + nx^2$. Then $\phi(0)=0$ and

$$\phi'(x) = n(1-x^2)^{n-1}\cdot 2x + 2nx \geq 0$$



$\phi'(x) > 0$ if $x\in(0,1)$.

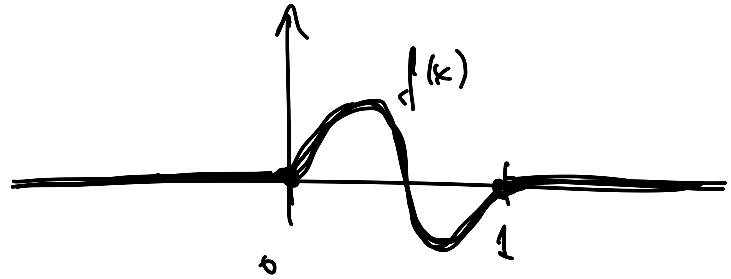
So $\phi(x)$ is strictly increasing, hence $\phi(x)>0$, $\forall x\in(0,1)$. $\square$

Pf of Weierstrass Approx. Thm. Without loss of generality assume $[a,b]=[0,1]$ and $f(0)=0=f(1)$, since:

- Translations and homotheties transform polynomials into polynomials and do not affect uniform convergence
- Consider $g(x) = f(x) - f(0) - x(f(1)-f(0))$
  $g(0) = 0 = g(1)$. If we succeed in approx. $g(x)$ with polynomials, then we will also be able to approx. $f(x)$ with polynomials. ← b/c $g(x)-f(x)$ is an affine function of x.

We may also extend $f:[0,1]\to\mathbb{R}$ continuously to $f:\mathbb{R}\to\mathbb{R}$ by setting $f(x)=0$ if $x\notin[0,1]$.

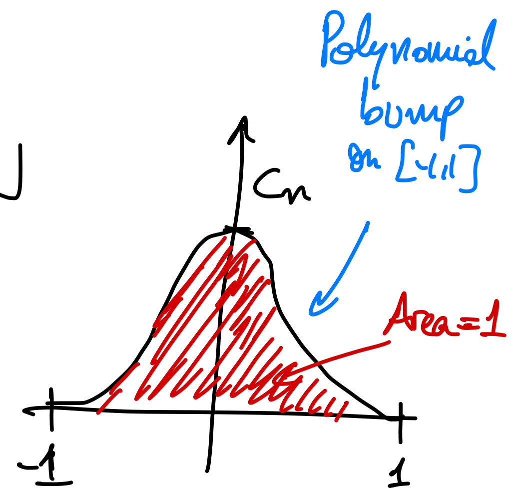
Since $f:[0,1]\to\mathbb{R}$ is continuous, we know it is uniformly continuous, and so is the extension $f:\mathbb{R}\to\mathbb{R}$.



Define $Q_n(x) = c_n(1-x^2)^n, \quad n\in\mathbb{N}$

where $c_n\in\mathbb{R}$ s.t.

$$\int_{-1}^{1} Q_n(x)\,dx = 1.$$



$Q_n(\pm 1) = 0$

Note that c_n grows in a controlled way:

$$\frac{1}{c_n} = \int_{-1}^{1}(1-x^2)^n\,dx = 2\int_0^1(1-x^2)^n\,dx \geq 2\int_0^{1/\sqrt{n}}(1-x^2)^n\,dx$$

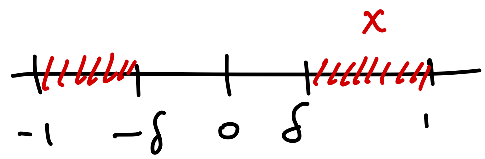
$$\overset{\text{Lemma}}{\geq} 2\int_0^{1/\sqrt{n}}(1-nx^2)\,dx = 2\left(x-\frac{nx^3}{3}\right)\Bigg|_0^{1/\sqrt{n}} =$$

$$= 2\left(\frac{1}{\sqrt{n}} - \frac{1}{3}\frac{1}{\sqrt{n}}\right) = 2\,\frac{2}{3\sqrt{n}} = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.$$

Thus $c_n < \sqrt{n}$, $\forall n \in \mathbb{N}$.

For $\delta > 0$, we therefore have: if $x \in [-1,1]$, $|x| \geq \delta$,

then $Q_n(x) = c_n(1-x^2)^n < \sqrt{n}(1-x^2)^n \leq \sqrt{n}(1-\delta^2)^n$



Now, define

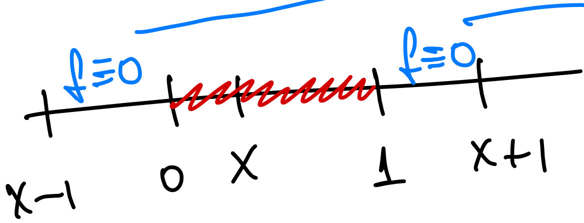
$$P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt$$

<u>Claim</u>: $P_n(x)$ is a polynomial in x.

Let $u = x+t$, so $du = dt$, and $\begin{cases} t = -1 \Leftrightarrow u = x-1 \\ t = +1 \Leftrightarrow u = x+1 \end{cases}$

$$\int_{-1}^{1} f(x+t)\, Q_n(t)\, dt = \int_{x-1}^{x+1} f(u)\, Q_n(u-x)\, du =$$

poly in x



$$= \underbrace{\int_0^1 f(u)\, Q_n(u-x)\, du}_{\text{Polynomial in } x}$$

(b/c we may move powers of x outside this integral in u)

$\square$

<u>Claim</u>: $P_n \to f$ uniformly.

Let $\varepsilon > 0$ be given, since $f: \mathbb{R} \to \mathbb{R}$ is uniformly cont.,

$\exists\, \delta > 0$ s.t. if $|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon/2$.

Since f is continuous on $[0,1]$ and $f \equiv 0$ outside $[0,1]$, so $f(\mathbb{R}) = f([0,1])$ is compact, hence bounded, i.e. $\exists M \in \mathbb{R}$ $f(\mathbb{R}) \subset [-M, M]$; i.e., $|f(x)| \le M,\ \forall x \in \mathbb{R}$

$$|P_n(x) - f(x)| = \Big|\underbrace{\int_{-1}^{1} f(x+t)\, Q_n(t)\, dt}_{P_n(x)} - \boxed{f(x)}\underbrace{\int_{-1}^{1} Q_n(t)\, dt}_{1}\Big|$$

$$= \left|\int_{-1}^{1} f(x+t)\, Q_n(t) - f(x)\, Q_n(t)\ dt\right|$$

$$= \left|\int_{-1}^{1} \big(f(x+t) - f(x)\big)\underbrace{Q_n(t)}_{\geq 0}\ dt\right|$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\ dt$$

Split integral $[-1,-\delta] \cup [-\delta,\delta] \cup [\delta,1]$

$$= \int_{-1}^{-\delta} \underbrace{|f(x+t) - f(x)|}_{\le 2M} Q_n(t)\, dt + \int_{-\delta}^{\delta} \underbrace{|f(x+t) - f(x)|}_{< \frac{\varepsilon}{2}} Q_n(t)\, dt$$

b/c $|x+t-x| = |t| < \delta$

$$+ \int_{\delta}^{1} \underbrace{|f(x+t) - f(x)|}_{\le 2M} Q_n(t)\, dt$$

$|f(x+t) - f(x)| \le |f(x+t)| + |f(x)| \le M + M = 2M$

$$\leq 2M\int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2}\underbrace{\int_{-\delta}^{\delta} Q_n(t)\,dt}_{\leq 1} + 2M\int_{\delta}^{1} Q_n(t)\,dt$$

$$\leq 2M\sqrt{n}\left(1-\delta^2\right)^n + \frac{\varepsilon}{2} + 2M\sqrt{n}\left(1-\delta^2\right)^n$$

$$= \underbrace{4M\sqrt{n}\left(1-\delta^2\right)^n}_{\downarrow \atop 0 \text{ as } n\nearrow+\infty} + \frac{\varepsilon}{2} < \varepsilon.$$

(b/c $(1-\delta^2)^n \to 0$ faster than $\sqrt{n} \to +\infty$)

for $n \in \mathbb{N}$ suff. large.

Therefore $P_n \to f$ uniformly.

$\square$

Stone - Weierstrass Approximation Theorem: ← What about polynomials makes the Weierstrass Approx. thm work?

Def: A *function algebra* is a subset $\mathcal{A}$ of the metric space $\mathcal{C}(M,\mathbb{R})$ of continuous functions $f: M \to \mathbb{R}$, such that:

i) $f, g \in \mathcal{A} \Rightarrow f+g,\ f\cdot g \in \mathcal{A}$ closed under $+$ and $\cdot$

ii) $c \in \mathbb{R},\ f \in \mathcal{A} \Rightarrow c\cdot f \in \mathcal{A}$ closed under mult. by scalars.

Def: A function algebra $A \subset \mathcal{C}(M,\mathbb{R})$ **separates points** if $\forall\, x,y \in M$, $x \neq y$ $\exists f \in A$ $f(x) \neq f(y)$; and A **vanishes nowhere** if $\forall x \in M$ $\exists f \in A$ s.t. $f(x) \neq 0$.

Stone-Weierstrass Approx. Thm. Suppose M is a **compact** metric space, and $A \subset \mathcal{C}(M,\mathbb{R})$ is a function algebra that separates points and vanishes nowhere. Then A is dense in $\mathcal{C}(M,\mathbb{R})$.

(In other words: $\forall f: M \to \mathbb{R}$ cont., $\exists\, p_n \in A$ s.t. $p_n \to f$ uniformly)

Note: Setting $M = [a,b]$, $A = \{p: [a,b] \to \mathbb{R}$ polynomials$\}$ one recovers the Weierstrass Approx. Thm.

Indeed, $p, q \in A \implies p+q \in A$, $p \cdot q \in A$

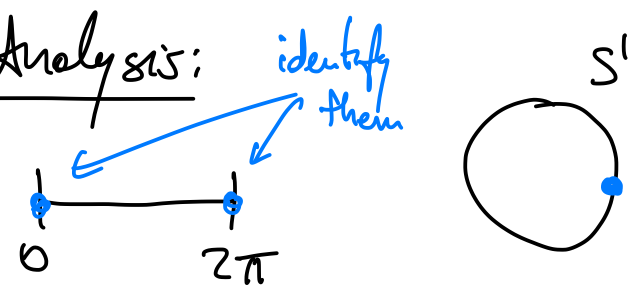
$c \cdot p \in A$, $\forall c \in \mathbb{R}$

Given $r, s \in [a,b]$, $r \neq s$, $p(x) = x$ separates them!

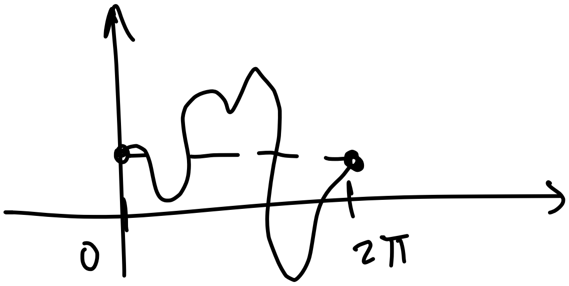
Moreover $\forall r \in [a,b]$, $p(x) = 1 \neq 0$.

An application to Fourier Analysis:

$M = S^1 = [0, 2\pi] / 0 \sim 2\pi$



$$\mathcal{C}(S^1,\mathbb{R}) = \{ f : S^1 \xrightarrow[\text{cont}]{} \mathbb{R} \} \cong \{ f : [0,2\pi] \to \mathbb{R} \text{ cont.}, \ f(0) = f(2\pi) \}.$$



$A \subset \mathcal{C}(S^1,\mathbb{R})$ trigonometric functions

$$A = \left\{ \sum_{k=0}^{+\infty} a_k \cos(kx) + \sum_{k=0}^{+\infty} b_k \sin(kx), \ a_k, b_k \in \mathbb{R} \right\}$$

→ A is a function algebra

→ A vanishes nowhere b/c $f(x) = a_0 \in A$, $\forall a_0 \in \mathbb{R}$

A separates points: indeed, if $x \neq y$, $x, y \in [0, 2\pi]$, then $\sin x \neq \sin y$ or $\cos x \neq \cos y$ unless $\{x,y\} = \{0, 2\pi\}$ ← these are identified in $S^1 = [0,2\pi]/_{0\sim 2\pi}$

By Stone–Weierstrass Theorem, A is dense in $\mathcal{C}(S^1,\mathbb{R})$

Ex: $A_{even} = \{ p : [-1,1] \to \mathbb{R}$ polynomial, $p(x) = p(-x), \ \forall x \}$

A_{even} is a function algebra, it vanishes nowhere, but it does not <u>separate points</u>.