Bounded sequences of functions:

Def: $f_n : E \to \mathbb{R}$ is pointwise bounded on E if

$$\forall x \in E, \quad \exists \phi(x) \in \mathbb{R} \quad \text{s.t.} \quad |f_n(x)| < \phi(x), \quad \forall n \in \mathbb{N}.$$

The bound depends on the point x.

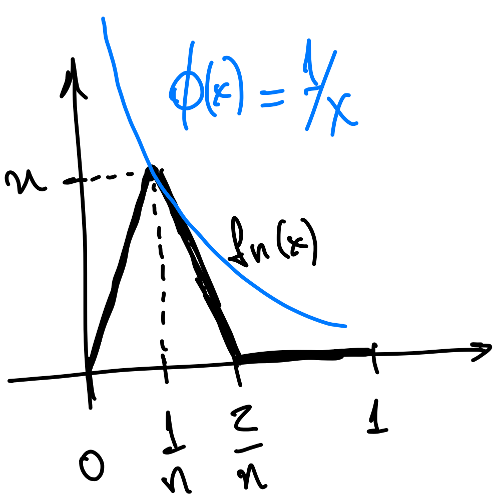
$f_n : E \to \mathbb{R}$ is uniformly bounded if $\exists M \in \mathbb{R}$ s.t.

$$|f_n(x)| < M, \quad \forall x \in E, \quad \forall n \in \mathbb{N}.$$

The bound does not depend on x.

Of course: unif. bounded $\Longrightarrow$ pointwise bounded.

But the converse does not hold; e.g.:



$$f_n : [0,1] \to \mathbb{R}$$

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}] \\ n^2\left(\frac{2}{n} - x\right) & \text{if } x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{otherwise} \end{cases}$$

- $f_n(x)$ is not unif. bounded: $\forall M \in \mathbb{R} \ \exists n \in \mathbb{N} \ n > M$

  so $f_n\left(\frac{1}{n}\right) = n > M$.

- $f_n(x)$ is pointwise bounded: $f_n(x) \le \phi(x) = \frac{1}{x}, \ \forall n \in \mathbb{N}$

<u>Note</u>: Even if $\{f_n : E \to \mathbb{R}\}$ is unif. bounded, there might not be any subsequence of $\{f_n : E \to \mathbb{R}\}$ that converges pointwise on E.

<u>Ex</u>: $f_n(x) = \sin nx$, $n \in \mathbb{N}$, $x \in [0, 2\pi]$.

<u>Claim</u>: There is no subsequence of $\{f_n\}$ that converges pointwise on $[0, 2\pi]$.

If not, say $f_{n_k}(x) = \sin(n_k x)$ converges pointwise on $[0, 2\pi]$.

Then:
$$\lim_{k \to \infty} \sin(n_k x) - \sin(n_{k+1} x) = 0 \quad \forall x \in [0, 2\pi]$$

So
$$\lim_{k \to \infty} \left(\sin(n_k x) - \sin(n_{k+1} x)\right)^2 = 0$$

By Lebesgue's theorem on integration:

We have not discussed this, but you won't have to use it on your own.

$$\lim_{k \to \infty} \int_0^{2\pi} \left(\sin(n_k x) - \sin(n_{k+1} x)\right)^2 dx = 0$$

However, $\int_0^{2\pi} \left(\sin(n_k x) - \sin(n_{k+1} x)\right)^2 dx = 2\pi$; hence we have the desired contradiction.

<u>Thm.</u> If $\{f_n : E \to \mathbb{R}\}$ is pointwise bounded and <u>E is countable</u>, then there exists a subsequence $\{f_{n_k} : E \to \mathbb{R}\}$ that converges pointwise.

Pf: Since E is countable, let $E = \{x_1, x_2, \dots, x_n, \dots\}$.

As $\{f_n(x_1)\}$ is bounded, there exists a convergent subsequence $\{f_{1,k}\}$ i.e. $f_{1,k}(x_1)$ converges as $k \nearrow +\infty$.

We can thus construct analogously a sequence of sequences:

$$
\begin{array}{llll}
S_1: & f_{1,1}, & f_{1,2}, & f_{1,3}, \dots \\
S_2: & f_{2,1}, & f_{2,2}, & f_{2,3}, \dots \\
S_3: & f_{3,1}, & f_{3,2}, & f_{3,3}, \dots \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

such that:

a) S_n is a subsequence of S_{n-1}, $\forall n = 2, 3, \dots$

b) $\{f_{n,k}(x_n)\}$ converges as $k \to \infty$ (This is possible b/c $\{f_n(x_j)\}$ is bounded.)

c) The order in which functions appear in each sequence must remain the same; if $f_{a,b}$ appears before $f_{a,c}$ in S_a, then the same happens in all subsequent S_n's, as long as both functions are there.

Consider the diagonal subsequence $f_{1,1}, f_{2,2}, f_{3,3}, \ldots$

By c), this sequence is a subsequence of S_n for all $n \in \mathbb{N}$, so b), it converges, i.e.,

$$f_{n,n}(x_i) \text{ converges } \forall i \in \mathbb{N} \text{ as } n \to \infty.$$

$\square$

<u>Def</u>: A family $\mathcal{F} = \{f_\lambda : E \to \mathbb{R}\}$ is <u>equicontinuous</u> on E if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall f \in \mathcal{F}$

$$d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon.$$

<u>Rmk</u>: If $\mathcal{F}$ is equicont., then every $f \in \mathcal{F}$ is <u>unif. cont.</u>

<u>Thm</u>. If K is compact and $f_n \in \mathcal{C}(K, \mathbb{R})$, $\forall n \in \mathbb{N}$ and f_n converges uniformly on K, then $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is equicontinuous.

<u>Pf</u>: Given $\varepsilon > 0$, since f_n converges uniformly, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow \|f_n - f_N\| < \varepsilon$. Recall that cont. functions on a compact set K are unif. continuous, hence $\exists \delta > 0$ s.t. $d(x,y) < \delta$, $1 \leq i \leq N$

$$\Rightarrow |f_i(x) - f_i(y)| < \varepsilon.$$

If $n > N$ and $d(x,y) < \delta$, altogether, we have:

$$|f_n(x)-f_n(y)| \le \underbrace{|f_n(x)-f_N(x)|}_{<\varepsilon} + \underbrace{|f_N(x)-f_N(y)|}_{<\varepsilon} + \underbrace{|f_N(y)-f_n(y)|}_{<\varepsilon}$$

$$\le 3\varepsilon.$$

Since ε is arbitrary, this implies $\mathcal{F}=\{f_n\}$ is equicontinuous. $\square$

<u>Arzelà–Ascoli Theorem</u>. If K is compact and $f_n \in \mathcal{C}(K,\mathbb{R})$, $\forall n\in\mathbb{N}$, and $\mathcal{F}=\{f_n\}$ is pointwise bounded and equicontinuous on K, then:

a) $\{f_n\}$ is uniformly bounded on K

b) $\{f_n\}$ has a subsequence that converges uniformly.

<u>Pf</u>: a) Given $\varepsilon>0$, since $\{f_n\}$ is equicont., $\exists\delta>0$ s.t.

$$d(x,y)<\delta \implies |f_n(x)-f_n(y)|<\varepsilon, \quad \forall n\in\mathbb{N}.$$

Since K is compact, there are finitely points $p_1, p_2, \dots, p_r \in K$ s.t. $\bigcup_{j=1}^{r} B(p_j,\delta) = K$. In other words, $\forall x\in K$, $\exists p_j$, $1\le j\le r$ s.t. $d(x,p_j)<\delta$. As $\{f_n\}$ is pointwise bounded, $\forall j=1,\dots,r$, $\exists M_j\in\mathbb{R}$ s.t. $|f_n(p_j)|<M_j$. Since there are only finitely many M_j's, let $M=\max\{M_j;\ j=1,\dots,r\}<+\infty$.

Thus, $|f_n(x)| = |f_n(x) - f_n(p_j) + f_n(p_j)|$

$$\le \underbrace{|f_n(x) - f_n(p_j)|}_{<\varepsilon} + \underbrace{|f_n(p_j)|}_{<M_j \le M} < M + \varepsilon.$$

Therefore $\{f_n\}$ is unif. bounded on K.

b) Let $E \subset K$ be a countable dense subset. By Thm above, $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ s.t. $\{f_{n_i}(x)\}$ converges (pointwise) for all $x \in E$. Let's simplify notation and write $g_i = f_{n_i}$.

Existence of such E is an exercise.

Claim: $\{g_i\}$ converges uniformly on K.

Given $\varepsilon > 0$, choose $\delta > 0$ by equicontinuity (as above).

Let $V(x,\delta) = \{y \in K : d(x,y) < \delta\}$. Since E is dense in K, $\exists\, x_1, \dots, x_m \in E$ s.t. $\bigcup_{s=1}^{m} V(x_s,\delta) \supset K$. Since $\{g_i(x)\}$ converges $\forall x \in E$, $\exists N \in \mathbb{N}$ s.t. $i, j \geq N$, $1 \le s \le m$

$$|g_i(x_s) - g_j(x_s)| < \varepsilon.$$

For any $x \in K$, $x \in V(x_s, \delta)$ for some $1 \le s \le m$. Therefore,

$$|g_i(x) - g_i(x_s)| < \varepsilon, \qquad \forall i \in N$$

If $i, j \geq N$ we have:

$$|g_i(x) - g_j(x)| \leq \underbrace{|g_i(x) - g_i(x_s)|}_{<\varepsilon} + \underbrace{|g_i(x_s) - g_j(x_s)|}_{<\varepsilon} - \underbrace{|g_j(x_s) - g_j(x)|}_{<\varepsilon}$$

$$\leq 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\{g_i\}$ converges uniformly on K. $\square$

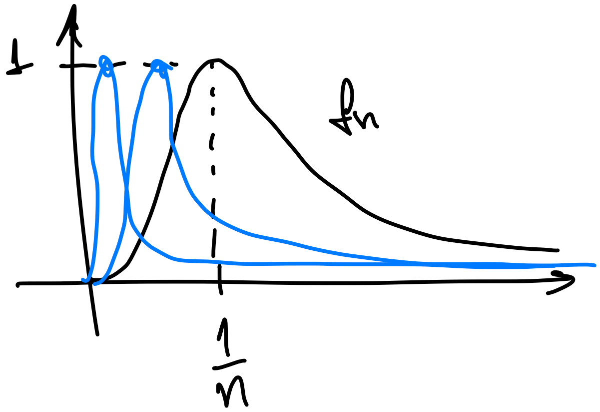
Examples: $f_n : [0,1] \to \mathbb{R}$. $n \in \mathbb{N}$

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}$$

- $\{f_n\}$ is uniformly bounded b/c:
  $|f_n(x)| \leq 1, \quad \forall x \in [0,1]$

- $\lim_{n \to \infty} f_n(x) = 0$, i.e., $\{f_n\}$ converges (pointwise) to 0.
  however $f_n\left(\frac{1}{n}\right) = \dfrac{(1/n)^2}{(1/n)^2 + (\underbrace{1 - n\,1/n}_{=0})^2} = 1.$

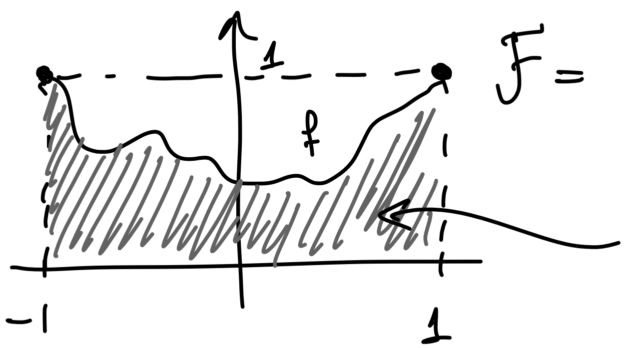
Therefore $\{f_n\}$ does not converge uniformly to 0.



Even though f_n converges (pointwise) to 0, the graphs do not converge, so the convergence is not uniform. (even for subsequences)

Thus, by Arzelà–Ascoli, we know $\mathcal{F} = \{f_n\}$ is <u>not</u> equicontinuous.

<u>What is the least area?</u> "Calculus of Variations"



$$\mathcal{F} = \left\{ f : [-1,1] \to [0,1],\ \text{continuous s.t. } f(-1) = 1 = f(1) \right\}$$

$$A(f) = \int_{-1}^{1} f(x)\,dx \qquad \text{"}\min_{f\in\mathcal{F}} A(f) = ?\text{"}$$

<u>Q</u>: Does there exist $f_0 \in \mathcal{F}$ s.t. $A(f) \geq A(f_0)$ for all $f \in \mathcal{F}$?

<u>A</u>: No: there is no such $f_0 \in \mathcal{F}$.

For any $f \in \mathcal{F}$, $A(f) > 0$. ⊛

$\forall n \in \mathbb{N}$, consider $f_n(x) = x^{2n}$. Clearly $f_n \in \mathcal{F}$.

$$A(f_n) = \int_{-1}^{1} x^{2n}\,dx = \frac{x^{2n+1}}{2n+1}\Big|_{-1}^{1} = \frac{2}{2n+1} \xrightarrow{n\nearrow+\infty} 0.$$

So if $f_0 \in \mathcal{F}$ existed, $A(f_0) \le A(f_n) = \frac{2}{2n+1}$

so $A(f_0) = 0$. This contradiction implies that no such $f_0 \in \mathcal{F}$ exists.

Rmk: $\mathcal{F}$ is not equicontinuous.

Indeed, if $\mathcal{F}$ was equicontinuous, then $\{f_n(x) = x^{2n}\}$ would also be equicontinuous; and hence by Arzelà–Ascoli, it would have a unif. conv. subsequence.

Q': Is there a fix?

A': Yes: consider the following subclass of $\mathcal{F}$:

$$\mathcal{F}_c = \left\{ f : [-1,1] \to [0,1] \, ; \, f(-1) = 1 = f(1) \atop |f(x) - f(y)| \le c|x-y|, \ \forall x,y \in [-1,1] \right\}$$

Lipschitz w/ const. c.

Note: $\forall c > 0$, $\mathcal{F}_c \subsetneq \mathcal{F}$

Claim: $\mathcal{F}_c$ is equicontinuous.

$\forall \varepsilon > 0$, let $\delta = \varepsilon / c$. Then if $x, y \in [-1,1]$

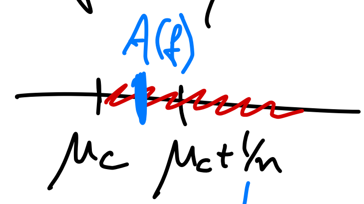
$$|x-y| < \delta \implies |f(x) - f(y)| \le c|x-y| < c \cdot \delta = \varepsilon.$$

Clearly, $\mathcal{F}_c$ is uniformly bounded:

$$|f(x)| \leq 1 \quad \forall x \in [-1,1], \quad \forall f \in \mathcal{F}_c$$

By Arzelà–Ascoli, any sequence of functions in $\mathcal{F}_c$ has a subsequence that converges uniformly on $[-1,1]$.

Let $\mu_c = \inf\{A(f) : f \in \mathcal{F}_c\}$



- non-empty
- bounded from below by 0.

$\forall n \in \mathbb{N}$, $\exists f_n \in \mathcal{F}_c$ s.t.

$$\mu_c \leq A(f_n) \leq \mu_c + \frac{1}{n} \quad (*)$$

Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$ that converges uniformly, say $f_{n_k} \longrightarrow \phi_c : [-1,1] \to [0,1]$.

$$A(\phi_c) = \int_{-1}^{1} \phi_c(x)\,dx = \int_{-1}^{1} \lim_{k\to\infty} f_{n_k}(x)\,dx = \lim_{k\to\infty} \int_{-1}^{1} f_{n_k}(x)\,dx$$

$$= \lim_{k\to\infty} A(f_{n_k}) \overset{(*)}{=} \mu_c .$$

So we found a continuous function $\phi_c : [-1,1] \to [0,1]$ with $\phi_c(-1) = 1 = \phi_c(1)$ which attains the inf.; i.e., the "area under" ϕ_c is the least possible among the areas under functions in $\mathcal{F}_c$.

Rmk: What does $\phi_c : [-1,1] \to [0,1)$ look like?