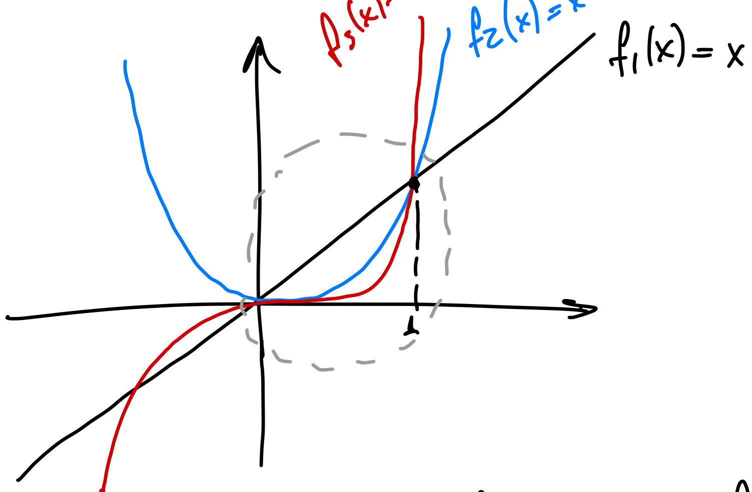
MAT320 Lecture 23 11/18/2020

Sequences and Series of functions

A sequence of functions $f_n : E \subset \mathbb{R} \to \mathbb{R}$ is a map from $\mathbb{N}$ to the set of functions from E to $\mathbb{R}$.

$$f_1 : E \to \mathbb{R}, \quad f_2 : E \to \mathbb{R}, \quad f_3 : E \to \mathbb{R}, \ \ldots \ f_n : E \to \mathbb{R}, \ldots$$

Ex: $f_n(x) = x^n$ $\quad (E = \mathbb{R})$

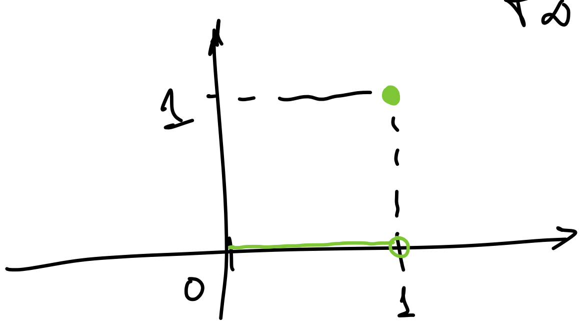


Note, e.g., if $x \in [0,1]$, we have:

$$f_n(x) = \begin{cases} x^n, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

$$f_n(x) < x, \quad \forall\, 0 < x < 1$$

$$f_\infty(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$



f_∞ is not continuous at $x = 1$.

Def: The sequence $\{f_n : E \to \mathbb{R}\}$ is said to converge pointwise to $f_\infty : E \to \mathbb{R}$ if $\forall x \in E$, $\lim_{n \to \infty} f_n(x) = f_\infty(x)$.

A <u>Series</u> of functions is a sequence $\{F_n\}$ of functions of the form $F_n(x) = \sum_{i=1}^{n} f_i(x) = f_1(x) + \dots + f_n(x)$ where $\{f_i : E \to \mathbb{R}\}$ is a sequence of functions.

"partial sum"

$$F_1(x) = f_1(x)$$
$$F_2(x) = f_1(x) + f_2(x)$$
$$F_3(x) = f_1(x) + f_2(x) + f_3(x)$$
$$\vdots$$
$$F_n(x) = f_1(x) + \dots + f_n(x)$$

If $\{F_n(x)\}$ converges pointwise to $F(x)$, then we write

$$F(x) = \sum_{i=1}^{+\infty} f_i(x) = \lim_{n\to\infty} F_n(x)$$

Can we exchange the order of the limits?

$$\lim_{m\to\infty} \lim_{n\to\infty} S_{m,n} \overset{?}{=} \lim_{n\to\infty} \lim_{m\to\infty} S_{m,n}$$

E.g., $S_{m,n} = \dfrac{m}{m+n}$, $m,n \in \mathbb{N}$

$$\lim_{m\to\infty} S_{m,n} = \lim_{m\to\infty} \frac{m^{\to\infty}}{m_{\to\infty} + n} = 1$$

$$\lim_{n\to\infty}\Big(\underbrace{\lim_{m\to\infty} s_{m,n}}_{1}\Big) = 1$$

$$\lim_{n\to\infty} s_{m,n} = \lim_{n\to\infty}\frac{m}{m+n} = 0 \qquad n \to \infty$$

$$\lim_{m\to\infty}\Big(\underbrace{\lim_{n\to\infty} s_{m,n}}_{=0}\Big) = 0$$

So, in general, the order in which we send the parameters to $+\infty$ will affect the limit. (i.e., may change the result!)

With the above point of view, we see that there is no reason to expect that the pointwise limit of a sequence of continuous functions is continuous:

$$\boxed{\varphi(x) \text{ is continuous at } x_0 \iff \lim_{x\to x_0}\varphi(x) = \varphi(x_0)}$$

If $f_n(x)$ is continuous at x_0, $\forall n\in\mathbb{N}$, i.e., $\lim_{x\to x_0} f_n(x) = f_n(x_0)$, then there is no reason to expect that $f_\infty(x)$ is:

not true in general

$$\lim_{x\to x_0} f_\infty(x) = \lim_{x\to x_0}\Big(\lim_{n\to\infty} f_n(x)\Big) \cancel{=} \lim_{n\to\infty}\Big(\underbrace{\lim_{x\to x_0} f_n(x)}_{f_n(x_0)}\Big) = f_\infty(x_0)$$

Examples:

1. $$f_n(x) = \frac{x^2}{(1+x^2)^n}, \qquad x\in\mathbb{R},\ n\in\mathbb{N}_0=\{0,1,2,\dots\}$$

$$F_n(x) = \sum_{i=0}^{n} f_i(x) = \sum_{i=0}^{n}\frac{x^2}{(1+x^2)^i} = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^n}$$

Let $F_\infty(x) = \lim_{n\to\infty} F_n(x) = \sum_{i=0}^{+\infty} f_i(x) = \sum_{i=0}^{+\infty} \frac{x^2}{(1+x^2)^i}$

If $x = 0$:

$$\left.\begin{aligned} & f_n(0) = \frac{0^2}{(1+0^2)^n} = 0, \quad \forall n \in \mathbb{N}_0 \\ & F_n(0) = \sum_{i=1}^{n} f_i(0) = 0 + \dots + 0 = 0, \quad \forall n \in \mathbb{N}_0 \\ & \lim_{n\to\infty} F_n(0) = 0. \end{aligned}\right\} F_\infty(0) = 0.$$

If $x \neq 0$:

$$F_n(x) = \sum_{i=0}^{n} \frac{x^2}{(1+x^2)^i} = x^2 \sum_{i=0}^{n} \frac{1}{(1+x^2)^i}$$

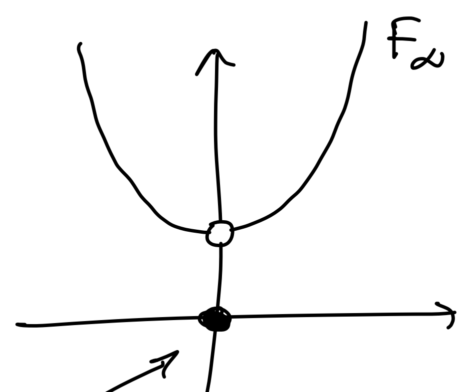
Geometric series with ratio $r = \frac{1}{1+x^2} < 1$ $\underline{\forall x \neq 0.}$

Thus $F_\infty(x) = x^2 \sum_{i=0}^{+\infty} \frac{1}{(1+x^2)^i} = x^2 \cdot \frac{1}{1 - \underbrace{\left(\frac{1}{1+x^2}\right)}_{r}} = x^2 \cdot \frac{1}{\left(\frac{\cancel{1} + x^2 - \cancel{1}}{1+x^2}\right)}$

$$= \frac{x^2(1+x^2)}{x^2} = 1 + x^2.$$

Altogether

$$F_\infty(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 + x^2 & \text{if } x \neq 0 \end{cases}$$

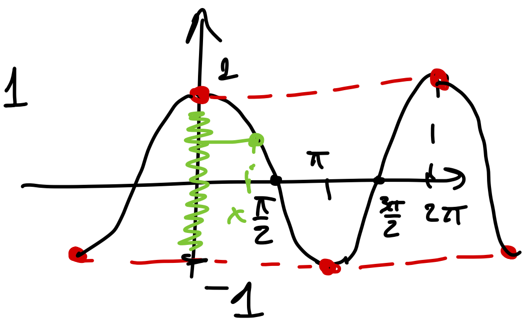


$\boxed{\text{discontinuous at } x = 0}$

(But, of course, each $F_n(x)$ $\underline{\text{is}}$ continuous at $x = 0$)

2. $f_m(x) := \lim_{n\to\infty} \left(\cos m!\pi x\right)^{2n}$ $\quad m = 1, 2, 3, \dots$

If $m!x \in \mathbb{Z}$, then $|\cos m!\pi x| = 1$



So $\underline{f_m(x) = 1}$

If $m!x \notin \mathbb{Z}$, then $|\cos m!\pi x| < 1$

So $\underline{f_m(x) = 0}$ $\quad$ b/c $\lim_{n\to\infty} c^n = 0$ if $|c| < 1$

$$f_m(x) = \begin{cases} 0 & \text{if } m!x \notin \mathbb{Z} \\ 1 & \text{if } m!x \in \mathbb{Z} \end{cases}$$

Let us now send $m \to \infty$, and write

$$f_\infty(x) = \lim_{m\to\infty} f_m(x).$$

$$\left(\underbrace{m!}_{\in \mathbb{N}} \cdot \underset{\notin \mathbb{Q}}{x} \notin \mathbb{Z}\right)$$

If $x \notin \mathbb{Q}$, then $f_m(x) = 0$, $\forall m \in \mathbb{N}$

so $f_\infty(x) = \lim_{m\to\infty} \underbrace{f_m(x)}_{= 0} = 0$

If $x \in \mathbb{Q}$, then $x = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ i.e.

if $m \geq q$ then $m!x = m!\frac{p}{q} \in \mathbb{Z}$, so $f_m(x) = 1$.

Thus $f_\infty(x) = \lim_{m\to\infty} \underbrace{f_m(x)}_{= 1 \text{ if } m \geq q} = 1.$

Altogether,

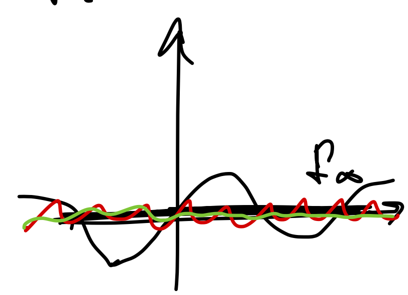
$$f_\infty(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases}$$

<u>Note</u>: $f_\infty(x)$ is everywhere discontinuous, even though $f_n(x)$ is continuous. Also $f_\infty(x)$ is <u>not</u> Riem.-integrable

3. $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$

$\forall x \in \mathbb{R}$, $|\sin nx| \leq 1$, so $|f_n(x)| \leq \dfrac{1}{\sqrt{n}} \xrightarrow{n\to\infty} 0$

$f_\infty(x) = \lim\limits_{n\to\infty} f_n(x) = 0$, $\forall x \in \mathbb{R}$



$$f_n'(x) = \frac{d}{dx}\frac{\sin nx}{\sqrt{n}} = \frac{1}{\sqrt{n}}\cos(nx)\cdot n$$

$$f_n'(x) = \sqrt{n}\cos(nx)$$

So, while $\{f_n\}$ converges pointwise to $f_\infty \equiv 0$, their derivatives $\{f_n'\}$ <u>do not</u> converge to $f_\infty' \equiv 0$.

$$f_n'(0) = \sqrt{n}\,\underbrace{\cos(n.0)}_{1} = \sqrt{n} \xrightarrow{n\nearrow\infty} +\infty.$$

So, in general, we <u>cannot</u> interchange (pointwise) limits of functions and derivatives:

not true in general

$$f_\infty'(x) = \frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) \neq \lim_{n\to\infty} f_n'(x)$$

4. $f_n(x) = n^2 x (1-x^2)^n \qquad 0 \le x \le 1, \quad n \in \mathbb{N}$

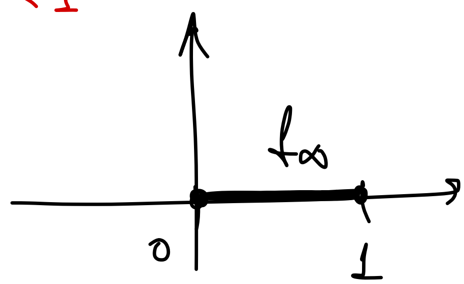
(Video 2 Lecture 10)

Let $f_\infty(x) = \lim_{n\to\infty} f_n(x)$.

If $0 < x \le 1$, then $\lim_{n\to\infty} n^2 x \underbrace{(1-x^2)}_{<1}{}^n = 0$.

If $x = 0$, $f_n(0) = 0$.

So $f_\infty(x) \equiv 0$.



But:

$$\int_0^1 f_n(x)\,dx = \int_0^1 n^2 x (1-x^2)^n dx = -n^2 \frac{(1-x^2)^{n+1}}{2(n+1)} \Bigg|_0^1 = \frac{n^2}{2n+2}$$

$$\int x(1-x^2)^n dx = \int u^n \frac{du}{-2} = -\frac{u^{n+1}}{2(n+1)}$$

$u = 1-x^2$
$du = -2x\,dx$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n^2}{2n+2} = +\infty.$$

<u>but</u>: $\int_0^1 f_\infty(x)\,dx = 0$

So, in general, one <u>cannot</u> interchange (pointwise) limits and integration:

not true in general

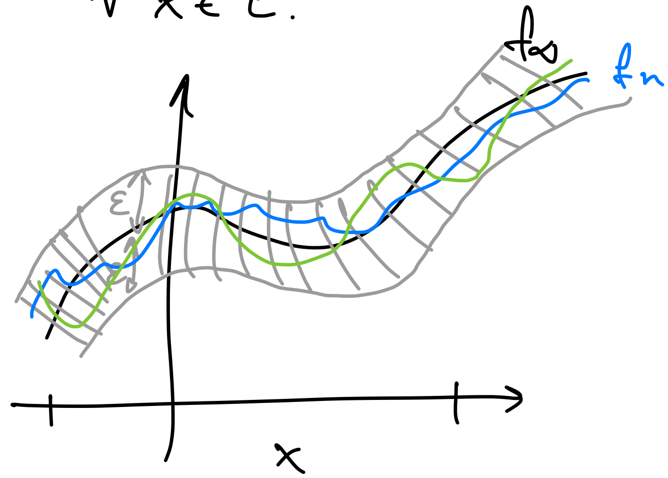
$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \underbrace{\lim_{n\to\infty} f_n(x)}_{f_\infty(x)}\,dx = \int_0^1 f_\infty(x)\,dx$$

Uniform convergence:

To try to mitigate the problems discussed above, we need a stronger notion of convergence. (Pointwise convergence is too weak)

Def: A sequence $\{f_n : E \to \mathbb{R}\}$ of functions converges uniformly to $f_\infty : E \to \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $|f_n(x) - f_\infty(x)| < \varepsilon \quad \forall x \in E$.

For n suff. large, the graph of $f_n(x)$ $\longrightarrow$ must be entirely contained in an ε-neighbd. of the graph of f_∞.



Comparison w/ pointwise convergence:

Pointwise: $f_n \xrightarrow[\text{pointwise}]{} f_\infty$

Here we may choose N using which x we want to look at.

$\forall x \in E, \ \forall \varepsilon > 0, \ \exists N = N(\varepsilon, x)$

s.t. $|f_n(x) - f_\infty(x)| < \varepsilon$

if $n \geq N$

Uniform convergence: $f_n \rightrightarrows f_\infty$

Here N does not depend on x: it must work for all $x \in E$.

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$

s.t. $|f_n(x) - f_\infty(x)| < \varepsilon, \forall x \in E$

if $n \geq N$.

We have the following Cauchy criterions

Thm. $f_n : E \to \mathbb{R}$ converges uniformly if and only if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. if $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq \varepsilon, \qquad \forall x \in E.$$

Pf: Given $\varepsilon > 0$ If $\{f_n\}$ converges unif. to $f_\infty : E \to \mathbb{R}$, then $\exists N \in \mathbb{N}$, $n \geq N \implies |f_n(x) - f_\infty(x)| \leq \frac{\varepsilon}{2}$.

So

$$|f_m(x) - f_n(x)| \overset{\text{Triangle ineq.}}{\leq} \underbrace{|f_m(x) - f_\infty(x)|}_{\leq \frac{\varepsilon}{2}} + \underbrace{|f_\infty(x) - f_n(x)|}_{\leq \frac{\varepsilon}{2}} \leq \varepsilon.$$

if $m, n \geq N$; $x \in E$.

Conversely, suppose $\forall \varepsilon > 0$ $\exists N$, $n, m \geq N \implies |f_n(x) - f_m(x)| \leq \varepsilon$ for all $x \in E$. Then $\{f_n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence, fixed which hence converges to a limit, which we call $f_\infty(x_0)$.

We claim that f_n converges unif. to f_∞.

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be s.t. $|f_n(x) - f_m(x)| \leq \varepsilon$ $\forall n, m \geq N$, $x \in E$. Then fix n and send $m \to \infty$:

$$\lim_{m \to \infty} \underbrace{|f_n(x) - f_m(x)|}_{\leq \varepsilon} \overset{\text{cont.}}{=} \Big| f_n(x) - \big(\underbrace{\lim_{m \to \infty} f_m(x)}_{f_\infty(x)}\big) \Big| = \underbrace{|f_n(x) - f_\infty(x)|}_{\leq \varepsilon}.$$

So $|f_n(x) - f_\infty(x)| \leq \varepsilon \quad \forall n \geq N$ and $x \in E$, i.e. $\square$

f_n converges uniformly to f_∞.

<u>Cor</u>: If $f_\infty(x) = \lim_{n\to\infty} f_n(x)$ is the pointwise limit of $f_n : E \to \mathbb{R}$, and $M_n = \sup_{x\in E} |f_n(x) - f_\infty(x)|$, then

$$f_n \text{ converges uniformly to } f_\infty \iff \lim_{n\to\infty} M_n = 0.$$

<u>Cor</u>: If $f_n : E \to \mathbb{R}$ satisfies $|f_n(x)| \leq M_n, \ \forall n \in \mathbb{N}$

then $\sum_{n=1}^{\infty} M_n < +\infty \implies \sum_{n=1}^{+\infty} f_n$ converges uniformly.

<u>Pf.</u> By Cauchy Criterion for series (of real numbers),

$\sum_{n=1}^{+\infty} M_n < +\infty \implies \forall \varepsilon > 0, \ \exists N \in \mathbb{N}$, if $m \geq n \geq N$ then

$$\sum_{i=n}^{m} M_i < \varepsilon$$

$$\left| \sum_{i=n}^{m} f_i(x) \right| \overset{\text{Triangle ineq}}{\leq} \sum_{i=n}^{m} \underbrace{|f_i(x)|}_{\leq M_i} \leq \sum_{i=n}^{m} M_i < \varepsilon$$

By Theorem above, the sequence $F_n(x) = \sum_{i=1}^{n} f_i(x)$ converges uniformly, i.e., $\sum_{i=1}^{+\infty} f_i(x)$ conv. unif. $\square$