Exercises / Review

Ex 1. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function which is periodic i.e., $\exists\, C > 0$ s.t. $f(x+C) = f(x)$, $\forall x \in \mathbb{R}$. Prove that $f(\mathbb{R})$ is a closed and bounded interval.

Pf: If $f(x) = c$, then $f(\mathbb{R}) = \{c\}$, which is a closed and bounded interval. Hence, assume f is not constant.

By induction, $f(x+nC) = f(x)$, $\forall n \in \mathbb{N}$:

$$f(x) = \underbrace{f(x+C)}_{n=1} = f(x+C+C) = f(x+2C+C) = \cdots$$

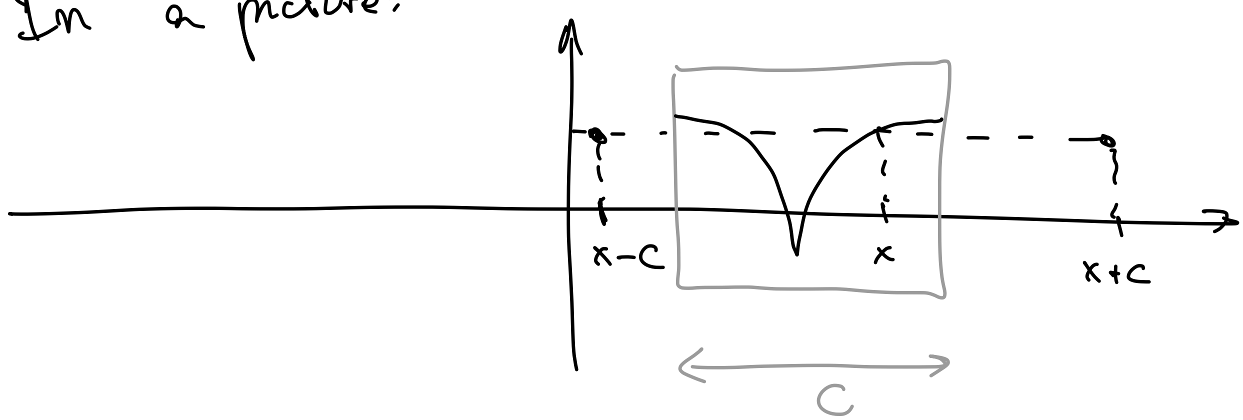
$$f(x+C+C) = \underbrace{f(x+2C)}_{n=2}, \qquad f(x+2C+C) = \underbrace{f(x+3C)}_{n=3}, \quad \ldots$$

Similarly, $f(x-nC) = f(x)$, $\forall n \in \mathbb{N}$:

$$f(x) = f(x-C+C) = \underbrace{f(x-C)}_{n=1} = f(x-2C+C) = \underbrace{f(x-2C)}_{n=2} = \cdots$$

Upshot: $f(x) = f(x+nC)$ $\forall n \in \mathbb{Z}$.

In a picture:



Claim: $\forall a \in \mathbb{R}\ \exists N \in \mathbb{Z}$ s.t. $a - NC \in [0, C]$.

Pf: Consider $E = \{n \in \mathbb{Z} : nC < a\} \subset \mathbb{Z}$

Then E is bounded from above by the Archimedean property: since $C > 0$, $\exists n \in \mathbb{N}$ s.t. $n.C > |a|$.

Let $N = \sup E$. Since E is closed and bounded, $N \in E$, so $N.C < a$, i.e., $a - NC \geq 0$. On the other hand $a - NC \leq C$, because otherwise:

$$a - NC > C \iff a > \underset{\substack{\Downarrow \\ N+1 \in E}}{(N+1)}C \quad \text{contradicting } N = \sup E$$

i.e. $a - NC \in [0, C]$

From the above claim: $f(\mathbb{R}) = f([0,C])$:

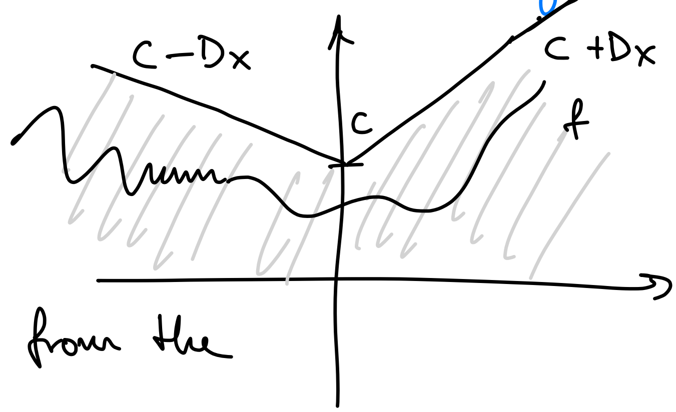
- $[0,C] \subset \mathbb{R} \implies f([0,C]) \subset f(\mathbb{R})$.
- To show the converse, suppose $b \in f(\mathbb{R})$, then $\exists a \in \mathbb{R}$ s.t. $f(a) = b$. Pick $N \in \mathbb{Z}$ as above, i.e., $\underline{a - NC} \in [0, C]$, then $f(\underline{a - NC}) = f(a) = b \in f([0,C])$.

Since $f: \mathbb{R} \to \mathbb{R}$ is continuous and $[0,C]$ is connected and compact, so is $f([0,C]) \subset \mathbb{R}$.
Since all connected subsets of $\mathbb{R}$ are intervals,

and $f([0,c])$ is compact, hence, by Heine-Borel Thm it is closed and bounded, it follows that $f([0,c])$ is a closed and bounded interval. $\square$

<u>Ex 2:</u> Suppose $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous. Prove that

$$\exists\, C, D > 0 \text{ s.t. } \underbrace{|f(x)| < C + D|x|,\quad \forall x \in \mathbb{R}.}_{\text{"}f(x)\text{ has at most linear growth"}}$$



<u>Pf.</u> Since f is unif. cont.

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$

Take $\varepsilon = 1$ and note that, from the above, we have $\exists\, \delta > 0$ s.t.

$$|x-y| < 2\delta \implies |f(x) - f(y)| < 1.$$

<u>Claim:</u> For all $k \in \mathbb{Z}$, if $|x| < k\cdot\delta$, then

$$|f(x)| < k + |f(0)|.$$

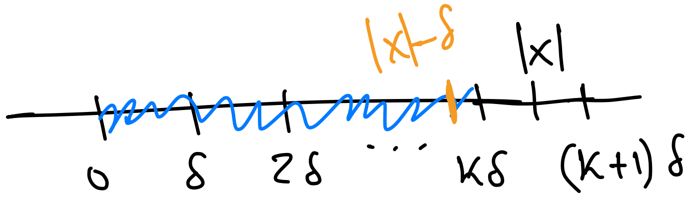
<u>Pf</u> (by induction on k). First: $k \geq 0$

$k=0$: $\quad |f(x)| < |f(0)|$ if $|x| < 0\cdot\delta = 0$ vacuously true

Induction step: Suppose if $|x| < k\delta$ then $|f(x)| < k + |f(0)|$

Let x be s.t. $|x| < (k+1)\delta$. WTS: $|f(x)| < k+1+|f(0)|$

Since $|x| < (k+1)\delta$



If $|x| < k\delta$, then apply induction hypothesis;

$$|f(x)| < k + |f(0)| < k+1+|f(0)|$$

If, instead, $k\delta \le |x| < (k+1)\delta$, then

$x > 0$ (analogous if $x<0$)

$$|f(x)| = |f(x) - f(x-\delta) + f(x-\delta)| \le \underbrace{|f(x) - f(x-\delta)|}_{<1} + \underbrace{|f(x-\delta)|}_{< k + |f(0)|}$$

$$\le 1 + k + |f(0)|$$

b/c $|x-(x-\delta)| = \delta < 2\delta$

by ind. hypothesis b/c $|x-\delta| < k\delta$

Do the induction for $k \le 0$ on your own! (This concludes proof of Claim).

Define $C := |f(0)| + 1$, $D := \frac{1}{\delta}$.

We claim that $\forall x \in \mathbb{R}$, $|f(x)| < C + D|x|$.

Indeed; set

$$m(x) = \min\{k \in \mathbb{N} : |x| < k\cdot\delta\}$$

note $m(x) \le \frac{|x|}{\delta} + 1$. Then:

$$|f(x)| \overset{\text{Claim}}{<} m(x) + |f(0)| \le \underbrace{\frac{|x|}{\delta}}_{D|x|} + \underbrace{1 + |f(0)|}_{C} = C + D|x|.$$

$\square$

Ex 3. Let $f:[a,b]\to\mathbb{R}$ be a function which is twice differentiable at $x_0\in(a,b)$. Define $g:[a,b]\to\mathbb{R}$ by

$$g(x)=\begin{cases}\dfrac{f(x)-f(x_0)}{x-x_0} & \text{if } x\neq x_0\\[2ex] f'(x_0) & \text{if } x=x_0\end{cases}$$

Prove g is differentiable at $x=x_0$ and compute $g'(x_0)$.

Compute $g'(x_0)$ using definition:

$$\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{\left(\left(\dfrac{f(x)-f(x_0)}{x-x_0}\right)-f'(x_0)\right)\cdot(x-x_0)}{(x-x_0)\cdot(x-x_0)}=$$

$$=\lim_{x\to x_0}\frac{f(x)-f(x_0)-(x-x_0)f'(x_0)}{(x-x_0)^2}=\dots$$

$f(x)-f(x_0)-(x-x_0)f'(x_0)\to 0$ as $x\to x_0$

$(x-x_0)^2\to 0$ as $x\to x_0$

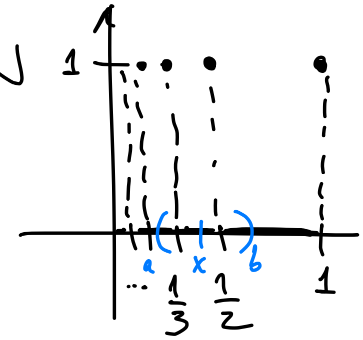
Check that L'Hospital's Rule can be applied:

$$\dots\overset{\text{L'H.}}{=}\lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{2(x-x_0)}=\frac{1}{2}\underbrace{\lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{x-x_0}}_{f''(x_0)}$$

$=\frac{1}{2}f''(x_0)$. This proves that g is differentiable at $x=x_0$, and $g'(x_0)=\frac{1}{2}f''(x_0)$. $\square$

Ex4. Let $f:[0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$



Prove that $f \in \mathcal{R}$ and compute $\int_0^1 f(x)\,dx$.

Note that if $0 < a < b < 1$, then $\exists\, x \in [a,b]$ s.t. $x \notin \{\frac{1}{n} : n \in \mathbb{N}\}$. (e.g., we may take $x \notin \mathbb{Q}$). Thus, since $f(x) = 0$ for such x, we have:

$$L(P,f) = 0 \quad \text{for all partitions } P \text{ of } [0,1],$$

b/c $m_i = 0$, $\forall i = 1, \ldots, n$.

Given $\varepsilon > 0$, let us find a partition P' s.t. $U(P',f) < 2\varepsilon$, which means that $f \in \mathcal{R}$ and $\int_0^1 f(x)\,dx = 0$.

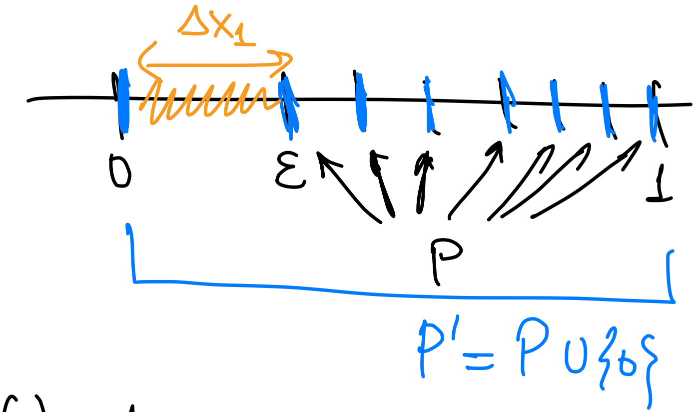
Let $g = f|_{[\varepsilon,1]}$, i.e., $g: [\varepsilon, 1] \to \mathbb{R}$, $g(x) = f(x)$, if $\varepsilon \leq x \leq 1$.

From Video 1 of Lecture 20, since g has only finitely many discontinuities, g is Riem.-integrable, $g \in \mathcal{R}$,

$\int_\varepsilon^1 g(x)\,dx = 0$ since $L(P,g) = 0$, $\forall P$, as above.

Since $g \in \mathcal{R}$, $\exists P$ of $[\varepsilon, 1]$ s.t. $U(P, g) < \varepsilon$.

Consider $P' = P \cup \{0\}$, which is a partition of $[0,1]$, satisfies:



$$U(P', f) = U(P, g) + \sup_{x \in [0,\varepsilon]} f(x) \cdot \underbrace{\Delta x_1}_{= \varepsilon}$$

$$< \varepsilon + 1 \cdot \varepsilon = 2\varepsilon.$$

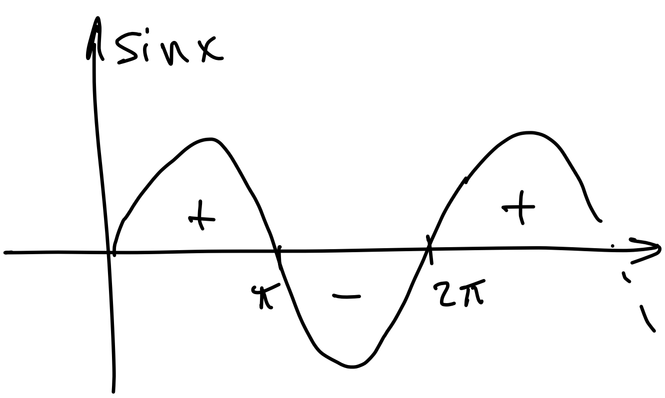
Therefore, $f \in \mathcal{R}$ and $\int_0^1 f(x)\,dx = 0$. $\square$

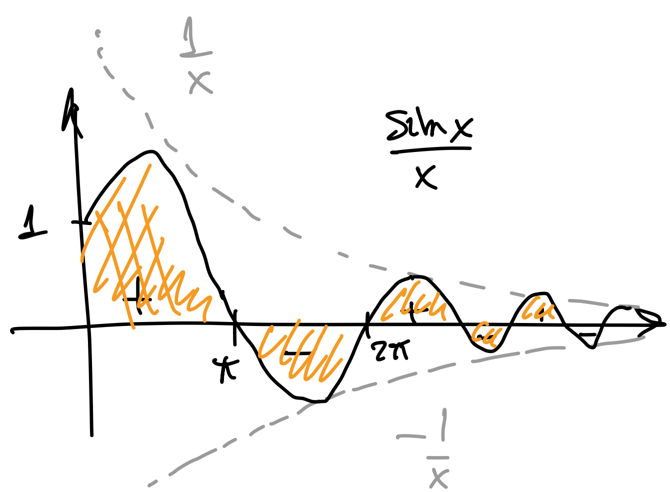
<u>Ex 5.</u> The improper integral $\int_a^{+\infty} f(x)\,dx$ is said to converge if $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists, and $\int_a^{+\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$.

Prove that $\int_0^{+\infty} \frac{\sin x}{x}\,dx$ converges.

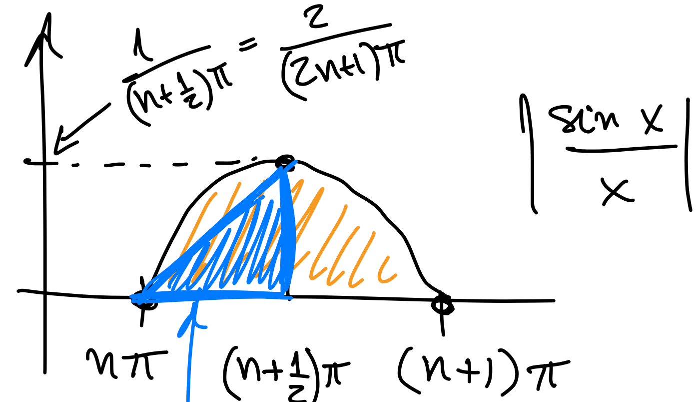
$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Does $\int_0^{+\infty} \left|\frac{\sin x}{x}\right| dx$ converge?





Let $a_n := \int_{n\pi}^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx$



Area $= \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{2}{(2n+1)\pi} = \frac{1}{2(2n+1)}$

Thus: $0 \le a_n \le \frac{2}{(2n+1)\pi} \cdot \pi$

$$0 \le a_n \le \frac{2}{2n+1}.$$

$$\int_0^{N\pi} \frac{\sin x}{x} dx = \underbrace{\int_0^{\pi} \frac{\sin x}{x} dx}_{a_0} + \underbrace{\int_{\pi}^{2\pi} \frac{\sin x}{x} dx}_{-a_1} + \dots + \int_{(N-1)\pi}^{N\pi} \frac{\sin x}{x} dx \quad \dots$$

$$= a_0 - a_1 + a_2 - a_3 + \dots \pm a_{N-1}$$

$$\int_0^{+\infty} \frac{\sin x}{x} dx = \lim_{N\to+\infty} \int_0^{N\pi} \frac{\sin x}{x} dx$$

$$= \lim_{N\to+\infty} \sum_{n=0}^{N-1} (-1)^n a_n < +\infty.$$

Alternating Series test

b/c

$a_n \geq 0$, a_n monot. decreasing, $a_n \to 0$.

$$\sum_{n=0}^{+\infty} (-1)^n a_n < +\infty. \qquad \left(a_n \le \frac{2}{2n+1}\right)$$

Thus: $\int_0^{+\infty} \frac{\sin x}{x} dx$ converges.

On the other hand, $\int_0^{+\infty} \left|\frac{\sin x}{x}\right| dx$ diverges.

Since $\dfrac{1}{2(2n+1)} \leq a_n, \quad \forall n \in \mathbb{N}$ (See above comparison with Area of triangle)

So $$\underbrace{\sum_{n=0}^{+\infty} \frac{1}{2(2n+1)}}_{+\infty} \leq \sum_{n=0}^{+\infty} a_n$$

Comparable to p-Series w/ $p=1$. (harmonic series)

$$\int_0^{+\infty} \left|\frac{\sin x}{x}\right| dx = \lim_{N \to +\infty} \int_0^{N\pi} \left|\frac{\sin x}{x}\right| dx$$

$$= \lim_{N \to +\infty} \sum_{n=0}^{N-1} a_n = +\infty .$$

$\square$

<u>Remark</u>. It can be shown that $\displaystyle\int_0^{+\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.