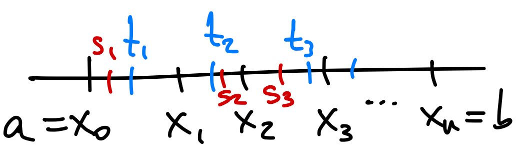
Integrating away the measure: Riemann-Stieltjes integral $\int f\,d\alpha$ $\rightsquigarrow$ Riemann integral $\int f\cdot\alpha'\,dx$

Thm: Assume $\alpha:[a,b]\to\mathbb{R}$ is monotonically increasing and $\alpha'\in\mathcal{R}$ on $[a,b]$. Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Then $f\in\mathcal{R}(\alpha) \iff f\cdot\alpha'\in\mathcal{R}$. In this case:

$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx$$

Pf: Let $\varepsilon>0$ be given. Since $\alpha'\in\mathcal{R}$, $\exists P=\{x_0\le x_1\le\dots\le x_n\}$ partition of $[a,b]$, s.t.

$$U(P,\alpha')-L(P,\alpha')\overset{(*)}{<}\varepsilon$$



By the Mean Value Thm, $\exists\, t_i\in[x_{i-1},x_i]$ s.t.

$$\Delta\alpha_i=\alpha(x_i)-\alpha(x_{i-1})=\alpha'(t_i)\,\underbrace{\Delta x_i}_{x_i-x_{i-1}}$$

Note that, setting $M=\sup_{x\in[a,b]}|f(x)|$, and $\underbrace{s_i\in[x_{i-1},x_i]}$, then: $\sum_{i=1}^n|\alpha'(t_i)-\alpha'(s_i)|\Delta x_i<\varepsilon$ $(*)$

$$\sum_{i=1}^n f(s_i)\Delta\alpha_i=\sum_{i=1}^n f(s_i)\,\alpha'(t_i)\Delta x_i$$

$$\left|\sum_{i=1}^n f(s_i)\Delta\alpha_i-\sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i\right|=\left|\sum_{i=1}^n f(s_i)\big(\alpha'(t_i)-\alpha'(s_i)\big)\Delta x_i\right|$$

$$\le M\cdot\varepsilon$$

In particular,

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\varepsilon$$

Since $s_i \in [x_{i-1}, x_i]$ can be chosen arbitrarily, it follows that

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon.$$

Reversing roles in the above,

$$U(P, f\alpha') \leq U(P, f, \alpha) + M\varepsilon$$

Altogether: $$\left| U(P, f, \alpha) - U(P, f\alpha') \right| \leq M\varepsilon$$

The above remains true for all refinements of P, so

$$\left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx \right| \leq M\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx.$$

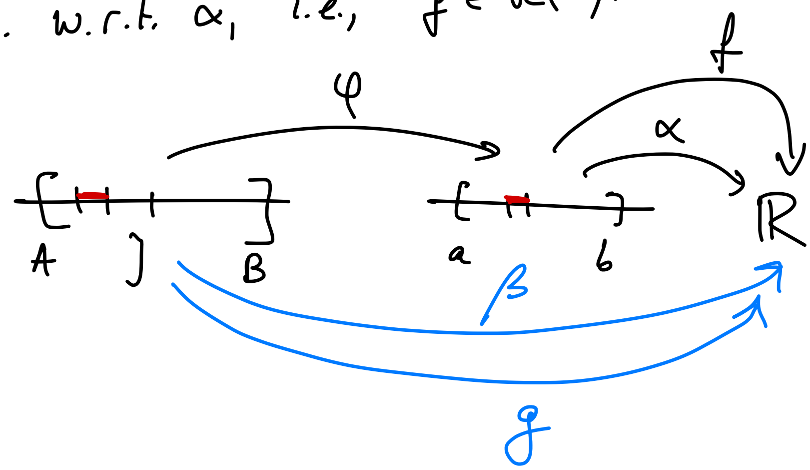
Repeating the above arguments for lower sums and lower integrals, one arrives at $\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\alpha'(x)\,dx$.

This establishes that $f \in \mathcal{R}(\alpha) \iff f\alpha' \in \mathcal{R}$ and $\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)\,dx$ in that case. $\square$

Change of variables formula

Thm. Suppose $\varphi : [A,B] \to [a,b]$ is strictly increasing and continuous, $\alpha : [a,b] \to \mathbb{R}$ monotonically increasing and $f : [a,b] \to \mathbb{R}$ is R.S.-int. w.r.t. α, i.e., $f \in \mathcal{R}(\alpha)$.

Let $\beta : [A,B] \to \mathbb{R}$ and
$g : [A,B] \to \mathbb{R}$ be defined:



$$\beta(y) = \alpha(\varphi(y))$$

$$g(y) = f(\varphi(y))$$

Then $g \in \mathcal{R}(\beta)$ and $\displaystyle\int_A^B g \, d\beta = \int_a^b f \, d\alpha$.

Pf: To each partition $P = \{x_0 \le x_1 \le \dots \le x_n\}$ of $[a,b]$ corresponds a partition $Q = \{y_0 \le y_1 \le \dots \le y_n\}$ of $[A,B]$, via $\varphi(y_i) = x_i$. All partitions of $[A,B]$ can be obtained in this way, since φ is increasing. The values assumed by f and g in corresponding intervals of the partitions P and Q are the same:

$$f([x_{i-1}, x_i]) = g([y_{i-1}, y_i])$$

Therefore:
$$U(Q, g, \beta) = U(P, f, \alpha)$$
$$L(Q, g, \beta) = L(P, f, \alpha)$$

Since $f \in \mathcal{R}(\alpha)$, both $U(P, f, \alpha)$ and $L(P, f, \alpha)$ can be made arbitrarily close to $\int f \, d\alpha$, hence also $U(Q, g, \beta)$ and $L(Q, g, \beta)$ can be made arbitrarily close to it. Thus $g \in \mathcal{R}(\beta)$ and $\int_A^B g \, d\beta = \int_a^b f \, d\alpha$. $\square$

Corollary: Taking $\alpha(x) = x$, we have $\beta(y) = \alpha(\varphi(y)) = \varphi(y)$:
If $\varphi' \in \mathcal{R}$ on $[A,B]$, then

$$\underbrace{\int_a^b f(x)\,dx}_{\int_a^b f\,d\alpha} = \underbrace{\int_A^B f(\varphi(y))\,\varphi'(y)\,dy}_{\int_A^B g\,d\beta}$$

Recall this is exactly the formula used in "u-substitution":

$x = \varphi(y)$
$dx = \varphi'(y)\,dy$

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(y))\,\varphi'(y)\,dy.$$

Integration and Differentiation.

Thm: Let $f : [a,b] \to \mathbb{R}$ be Riem.-integrable, i.e. $f \in \mathcal{R}$, and define

$$F : [a,b] \to \mathbb{R}, \qquad F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$, and if f is cont. at $x_0 \in [a,b]$ then $F'(x_0) = f(x_0)$.

Pf. Let $M \in \mathbb{R}$ be s.t. $|f(t)| \le M$ for all $t \in [a,b]$. If $a \le x < y \le b$, then

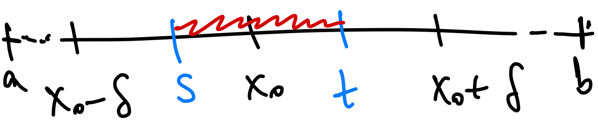
$$|F(y) - F(x)| = \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| = \left| \int_x^y f(t)\,dt \right| \le \underbrace{M \cdot (y-x)}_{< \varepsilon}$$

Given $\varepsilon > 0$, we have $|F(y) - F(x)| < \varepsilon$ if $|x-y| < \varepsilon/M$.
Therefore F is continuous (actually unif. cont.).

Suppose f is cont. at $x_0 \in [a,b]$, then $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$|t - x_0| < \delta \implies |f(t) - f(x_0)| < \varepsilon.$$

If $x_0 - \delta < s \le x_0 \le t < x_0 + \delta$ and $a \le s < t \le b$, then



$$\frac{F(t) - F(s)}{t-s} = \frac{1}{t-s}\Big(\underbrace{\int_a^t f(z)\,dz}_{F(t)} - \underbrace{\int_a^s f(z)\,dz}_{F(s)} \Big)$$

$$= \frac{1}{t-s}\int_s^t f(z)\,dz$$

So
$$\left| \frac{F(t)-F(s)}{t-s} - f(x_0) \right| = \left| \left(\frac{1}{t-s}\int_s^t f(z)\,dz \right) - f(x_0) \right|$$

$f(x_0) = \frac{1}{t-s}\int_s^t f(x_0)\,dz$

$$= \left| \frac{1}{t-s}\int_s^t \underbrace{\big(f(z) - f(x_0)\big)}_{<\varepsilon}\,dz \right|$$

$$< \frac{1}{t-s}\cdot (t-s) \cdot \varepsilon = \varepsilon.$$

Hence $F'(x_0) = f(x_0)$. $\square$

Fundamental Theorem of Calculus:

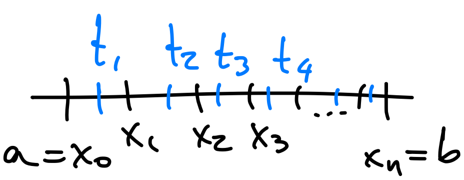
Thm. If $f:[a,b]\to\mathbb{R}$ is Riem.- integrable ($f\in\mathcal{R}$) and if $\exists F:[a,b]\to\mathbb{R}$ differentiable s.t. $F'(x)=f(x)$, $\forall x\in[a,b]$, then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Pf: Let $\varepsilon>0$ be given, and choose a partition $P=\{x_0\le x_1\le\ldots\le x_n\}$ of $[a,b]$ s.t. $U(P,f)-L(P,f)<\varepsilon$.

By the Mean Value Thm, $\exists t_i\in[x_{i-1},x_i]$

$$F(x_i)-F(x_{i-1}) = \underbrace{F'(t_i)}_{f(t_i)}\underbrace{(x_i-x_{i-1})}_{\Delta x_i}$$



Thus

$$\sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} F(x_i)-F(x_{i-1}) \overset{\text{telescopic sum}}{=} F(b)-F(a)$$

For the $\varepsilon>0$ given above, since $U(P,f)-L(P,f)<\varepsilon$, we have

$$\left|\sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f(x)\,dx\right| < \varepsilon$$

By triangle ineq:

$$\left|F(b)-F(a)-\int_a^b f(x)\,dx\right| < \varepsilon$$

Since $\varepsilon>0$ was arbitrary, it follows: $\int_a^b f(x)\,dx = F(b)-F(a)$. $\square$

Integration by Parts:

Thm. Suppose $F, G : [a,b] \to \mathbb{R}$ are differentiable, and $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$. Then

$$\int_a^b F(x)g(x)\,dx = \underbrace{F(b)G(b) - F(a)G(a)}_{F(x)G(x)\Big|_a^b} - \int_a^b f(x)G(x)\,dx$$

Pf: Set $H(x) = F(x)G(x)$ and apply the previous result:

$$h(x) = H'(x) = F'(x)G(x) + F(x)G'(x)$$

$$= \underbrace{f(x)}_{\in\mathcal{R}}\underbrace{G(x)}_{\in\mathcal{R}} + \underbrace{F(x)}_{\in\mathcal{R}}\underbrace{g(x)}_{\in\mathcal{R}} \in \mathcal{R}.$$

So $\underbrace{\int_a^b h(x)\,dx}_{\int_a^b F(x)g(x)\,dx + \int_a^b f(x)G(x)\,dx} = H(b) - H(a)$, i.e.

$$\int_a^b F(x)g(x)\,dx + \int_a^b f(x)G(x)\,dx = \underbrace{F(b)G(b)}_{H(b)} - \underbrace{F(a)G(a)}_{H(a)}$$

i.e. $\displaystyle\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$ $\square$

For vector-valued functions, $f : [a,b] \to \mathbb{R}^k$, define

$$\int_a^b f\,d\alpha = \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, \dots, \int_a^b f_k\,d\alpha\right), \text{ where } f = (f_1, f_2, \dots, f_k)$$

i.e., proceed coordinate-by-coordinate.