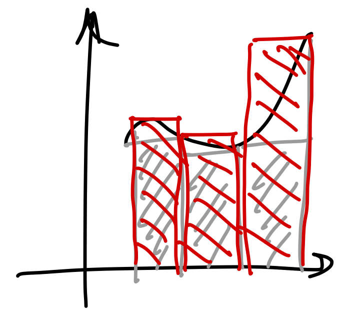
Riemann-Stieltjes integrability

Recall (Lecture 19, Video 6):



$f:[a,b]\to\mathbb{R}$ is R.S.-integrable w.r.t. α $\iff$ $\forall \varepsilon>0$, $\exists P$ partition of $[a,b]$ s.t. $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$

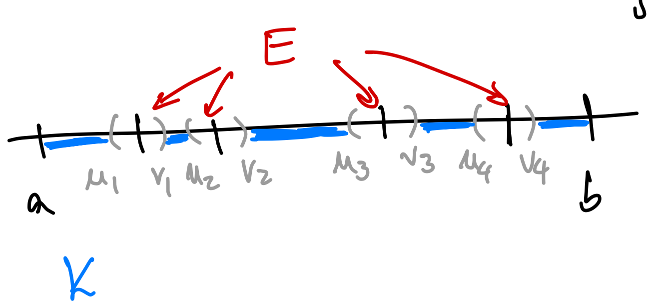
Upper and lower Riemann-Stieltjes sums of f w.r.t. α and partition P.

Thm. If $f:[a,b]\to\mathbb{R}$ is bounded and has only finitely many points of discontinuity, such that $\alpha:[a,b]\to\mathbb{R}$ is continuous at the discontinuities of f. Then f is R.S.-integrable w.r.t. α, $f\in\mathcal{R}(\alpha)$.

Pf. Let $\varepsilon>0$ be given. Set $M=\sup|f(x)|$ and

$E=\{x\in[a,b] : f \text{ is discontinuous at } x\}$.

Since $|E|<\infty$ and α is continuous at all $x\in E$, we may cover E with finitely many disjoint intervals $[u_j,v_j]\subset[a,b]$ s.t. $\sum_j \alpha(v_j)-\alpha(u_j)<\varepsilon$. Also, we may arrange $[u_j,v_j]$ so that each point of $E\cap(a,b)$ lies in the interior of some $[u_j,v_j]$.



Note that $K = [a,b] \setminus \bigcup_j (u_j, v_j)$ is compact. Thus, as f is continuous on K, it follows that f is uniformly continuous, i.e., $\exists \delta > 0$ s.t. $\forall s,t \in K$

$$|s-t| < \delta \implies |f(s) - f(t)| < \varepsilon.$$

Build a partition $P = \{a = x_0 \le x_1 \le \dots \le x_n = b\}$ as follows:

- $u_j, v_j \in P$, $\forall j$
- No point in (u_j, v_j) is in P
- If $x_{i-1} \notin \{u_j : j\}$, then $\Delta x_i = x_i - x_{i-1} < \delta$.

Now, $M_i - m_i \le 2M$ for all i

$M_i - m_i \le \varepsilon$ unless $x_i \in \{u_j : j\}$

Thus

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_i (M_i - m_i)\Delta\alpha_i$$

$\sum_i \Delta\alpha_i = \alpha(b) - \alpha(a)$ (telescopic)

can be made arbitrarily small.

$$\le \boxed{(\alpha(b) - \alpha(a)) \cdot \varepsilon + 2M\varepsilon} = \frac{\varepsilon'}{2} < \varepsilon'$$

Since $\varepsilon > 0$ was arbitrary, the above can be made $< \varepsilon'$ for all $\varepsilon' > 0$. $\square$

Thm. If $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \le f \le M$, $\phi : [m,M] \to \mathbb{R}$ is continuous, and $h(x) = \phi(f(x))$, then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

Pf: Given $\varepsilon > 0$, since ϕ is cont. on the compact interval $[m,M]$, it is unif. cont., hence $\exists \delta > 0$ s.t. $\delta < \varepsilon$ and $\forall s,t \in [m,M]$, $|s-t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \varepsilon$.

Since $f \in \mathcal{R}(\alpha)$, $\exists P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ s.t.

$$(*) \quad U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$$

Let M_i, m_i be the usual quantities for f, and M_i^*, m_i^* be the corresponding quantities for h.

Let
$$A = \{ i : 1 \le i \le n, \ M_i - m_i < \delta \}$$
$$B = \{ i : 1 \le i \le n, \ M_i - m_i \ge \delta \}$$

For $i \in A$, we have $M_i^* - m_i^* \le \varepsilon$

For $i \in B$, $M_i^* - m_i^* \le 2K$ where $K = \sup_{t \in [m,M]} |\phi(t)|$.

By $(*)$, we have:

$$\delta \cdot \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i) \Delta\alpha_i \overset{(*)}{<} \delta^2$$

So, dividing by $\delta > 0$, we conclude $\sum_{i \in B} \Delta\alpha_i < \delta$.

$$U(P,h,\alpha)-L(P,h,\alpha) = \sum_{i\in A} \underbrace{(M_i^* - m_i^*)}_{\le \varepsilon} \Delta\alpha_i + \sum_{i\in B} \underbrace{(M_i^* - m_i^*)}_{\le 2K} \Delta\alpha_i$$

$\sum_{i\in A} \Delta\alpha_i = \alpha(b)-\alpha(a)$

$$\le \varepsilon \cdot (\alpha(b)-\alpha(a)) + 2K\delta$$

$$\overset{\delta<\varepsilon}{<} \varepsilon\left(\alpha(b)-\alpha(a) + 2K\right)$$

Since the above can be made arbitrarily small, we conclude that h is R.S.- integrable: $h \in R(\alpha)$. □

Properties of the Riemann-Stieltjes integral

$\int f\,d\alpha \overset{\alpha(x)=x}{=} \int f\,dx$

1. $f_1, f_2 \in R(\alpha) \implies f_1 + f_2 \in R(\alpha)$

$c f_1 \in R(\alpha), \quad \forall c \in \mathbb{R}$

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$

$$\int_a^b c\cdot f_1\,d\alpha = c\int_a^b f_1\,d\alpha$$

Linearity of R.S.-int. on $\underline{\underline{f}}$ (integrand)

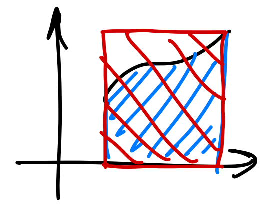
2. If $f_1 \le f_2$ on $[a,b]$, then $\int_a^b f_1\,d\alpha \le \int_a^b f_2\,d\alpha$

3. If $f \in R(\alpha)$ on $[a,b]$ and $a<c<b$, then $f\in R(\alpha)$ on $[a,c]$ and on $[c,b]$ and:

$$\int_a^c f\,d\alpha + \int_c^b f\,d\alpha = \int_a^b f\,d\alpha.$$

4. If $f\in R(\alpha)$ on $[a,b]$ and $|f(x)|\le M$ on $[a,b]$, then

$$\left|\int_a^b f\,d\alpha\right| \le M\cdot\left(\alpha(b)-\alpha(a)\right)$$



$\alpha(x)=x$

5. If $f\in\mathcal{R}(\alpha_1)\cap\mathcal{R}(\alpha_2)$, then $f\in\mathcal{R}(\alpha_1+\alpha_2)$ and $f\in\mathcal{R}(c\alpha_1)$ for any $c\in\mathbb{R}$, and:

$$\left.\begin{aligned} \int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2 &= \int_a^b f\,d(\alpha_1+\alpha_2)\\ c\cdot\int_a^b f\,d\alpha_1 &= \int_a^b f\,d(c\alpha_1)\end{aligned}\right\}$$

Linearity of R.S.-int. on α (measure)

<u>Pf.</u> 1) If P is a partition of $[a,b]$, then

$$\left.\begin{aligned} &L(P,f_1,\alpha)+L(P,f_2,\alpha)\le L(P,\underbrace{f_1+f_2}_{f},\alpha)\le U(P,\underbrace{f_1+f_2}_{f},\alpha)\\ &U(P,\underbrace{f_1+f_2}_{f},\alpha)\le U(P,f_1,\alpha)+U(P,f_2,\alpha)\end{aligned}\right\}(*)$$

If $f_1,f_2\in\mathcal{R}(\alpha)$, then $\forall\varepsilon>0$, $\exists\, P_1.\,P_2$ partitions

$$U(P_1,f_1,\alpha)-L(P_1,f_1,\alpha)<\varepsilon/2$$

$$U(P_2,f_2,\alpha)-L(P_2,f_2,\alpha)<\varepsilon/2$$

Letting $P=P_1\cup P_2$ be their common refinement, by $(*)$,

$$U(P,f_1+f_2,\alpha)-L(P,f_1+f_2,\alpha)\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Thus $f_1+f_2\in\mathcal{R}(\alpha)$. Moreover,

$$U(P, f_1, \alpha) < \int_a^b f_1 \, d\alpha + \varepsilon$$

$$U(P, f_2, \alpha) < \int_a^b f_2 \, d\alpha + \varepsilon$$

Using the above and inequalities $(*)$, writing $f = f_1 + f_2$,

$$\int_a^b f \, d\alpha \leq \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha + 2\varepsilon$$

Since the above holds $\forall \varepsilon > 0$, we have:

$$\int_a^b f \, d\alpha \leq \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha.$$

Replacing f_1, f_2 by $-f_1, -f_2$, one obtains the reverse ineq., so equality must hold. $\square$

Leave the other proofs as exercises.

<u>Thm</u>. If $f, g \in \mathcal{R}(\alpha)$ on $[a,b]$, then $f \cdot g \in \mathcal{R}(\alpha)$, $|f| \in \mathcal{R}(\alpha)$ and $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$.

<u>Pf</u>: Let $\phi(t) = t^2$. By a thm above, $\phi(f) = f^2 \in \mathcal{R}(\alpha)$, and $\phi(g) = g^2 \in \mathcal{R}(\alpha)$. Moreover $f+g \in \mathcal{R}(\alpha)$, $f-g \in \mathcal{R}(\alpha)$, so $\phi(f+g) = (f+g)^2 \in \mathcal{R}(\alpha)$, $\phi(f-g) = (f-g)^2 \in \mathcal{R}(\alpha)$.

$$f \cdot g = \frac{1}{4}\left(\underbrace{(f+g)^2}_{\in \mathcal{R}(\alpha)} - \underbrace{(f-g)^2}_{\in \mathcal{R}(\alpha)} \right) \in \mathcal{R}(\alpha).$$

! There is no general formula to compute $\int f \cdot g \, d\alpha$

Next, let $\phi(t) = |t|$; so by Thm above $\phi(f) = |f| \in \mathcal{R}(\alpha)$.

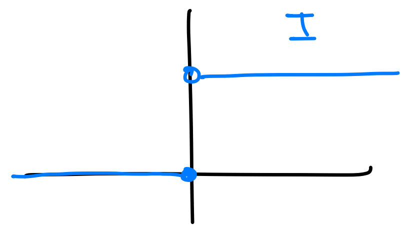
Let $c = \pm 1$ so that $c \int f\, d\alpha \geq 0$.

$cf \leq |f|$

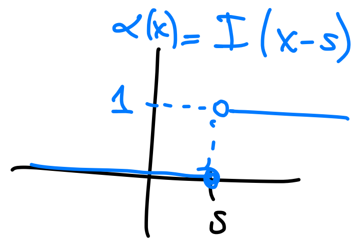
Then
$$\left| \int_a^b f\, d\alpha \right| = c \int_a^b f\, d\alpha = \int_a^b cf\, d\alpha \leq \int_a^b |f|\, d\alpha .$$

$\square$

<u>Definition</u>. The <u>unit step</u> function $I : \mathbb{R} \to \mathbb{R}$ is
$$I(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

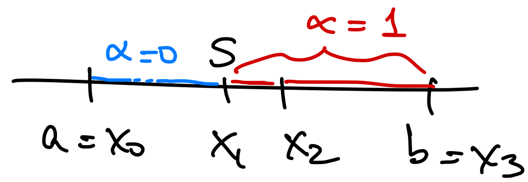


<u>Thm</u>. If $s \in (a,b)$, f is bounded $[a,b]$, f is cont. at s, then setting $\alpha(x) = I(x-s)$ we have
$$\int_a^b f\, d\alpha = f(s).$$



<u>Pf</u>: Consider a partition $P = \{x_0, x_1, x_2, x_3\}$

$x_1 = s < x_2 < x_3 = b$. Then



$U(P, f, \alpha) = M_2, \quad L(P, f, \alpha) = m_2$

Letting $x_2 \to s$, since f is cont. at s, we have that

$M_2, m_2 \longrightarrow f(s)$, so, in the limit, $\int_a^b f\, d\alpha = f(s)$.

$\square$

Thm. Suppose $c_n \geq 0$, $n \in \mathbb{N}$, $\sum c_n < +\infty$.

If $\{s_n\}$ is a seq. of distinct points in (a,b)

and $$\alpha(x) = \sum_{n=1}^{+\infty} c_n I(x-s_n).$$

Let $f:[a,b] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f\, d\alpha = \sum_{n=1}^{+\infty} c_n \cdot f(s_n)$$

Pf.: By comparison, $\sum c_n \underbrace{I(x-s_n)}_{0 \leq \cdots \leq 1} \leq \sum c_n < +\infty$ so

$\alpha(x)$ is well-def. Moreover $\alpha(a) = 0$, $\alpha(b) = \sum c_n$.

Let $\varepsilon > 0$ be given, choose $N \in \mathbb{N}$ so that

$$\sum_{N+1}^{+\infty} c_n < \varepsilon.$$

Let
$$\left.\begin{aligned} \alpha_1(x) &= \sum_{n=1}^{N} c_n I(x-s_n) \\ \alpha_2(x) &= \sum_{n=N+1}^{+\infty} c_n I(x-s_n) \end{aligned}\right\} \quad \alpha(x) = \alpha_1(x) + \alpha_2(x).$$

By properties of R.S.- integral,

$$\int_a^b f\, d\alpha_1 = \sum_{i=1}^{N} c_n f(s_n).$$

Since $\alpha_2(b) - \alpha_2(a) < \varepsilon$, $\left| \int_a^b f \, d\alpha_2 \right| \leq M \cdot \varepsilon$, where

$$M = \sup |f(x)|$$

Since $\alpha = \alpha_1 + \alpha_2$, we have:

$$\Bigg| \underbrace{\int_a^b f \, d\alpha}_{\int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2} - \sum_{n=1}^{N} c_n f(s_n) \Bigg| \leq M \cdot \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, letting $N \to +\infty$, one has

$$\int_a^b f \, d\alpha = \sum_{n=1}^{+\infty} c_n f(s_n).$$

$\square$