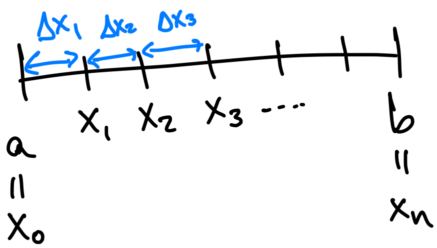
Riemann Integral

Def: Let $[a,b]$ be given. A <u>partition</u> P of $[a,b]$ is a finite subset



$$P = \{a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_{n-1} \leq x_n = b\}$$

We also define

$$\Delta x_i = x_i - x_{i-1} \qquad i = 1, 2, \dots, n$$

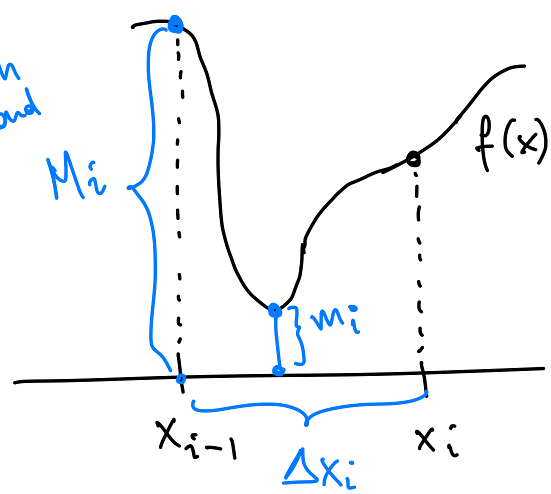
Let $f : [a,b] \to \mathbb{R}$, then we define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

Upper Riemann sum:

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

geometrically, this is the Area of the rectangle with height M_i and base Δx_i



Lower Riemann sum:

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

... height m_i base Δx_i

Finally, define the upper and lower Riemann integrals of $f : [a,b] \to \mathbb{R}$ on $[a,b]$ as:

$$\boxed{\overline{\int_a^b} f\,dx = \inf_P U(P,f), \qquad \underline{\int_a^b} f\,dx = \sup_P L(P,f)}$$

Def: $f:[a,b]\to\mathbb{R}$ is Riemann-integrable if $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$. In this case, we write

$$\int_a^b f\,dx := \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$$

and $f\in\mathcal{R}$.

$\mathcal{R}$ denotes the set of Riemann-integrable functions.

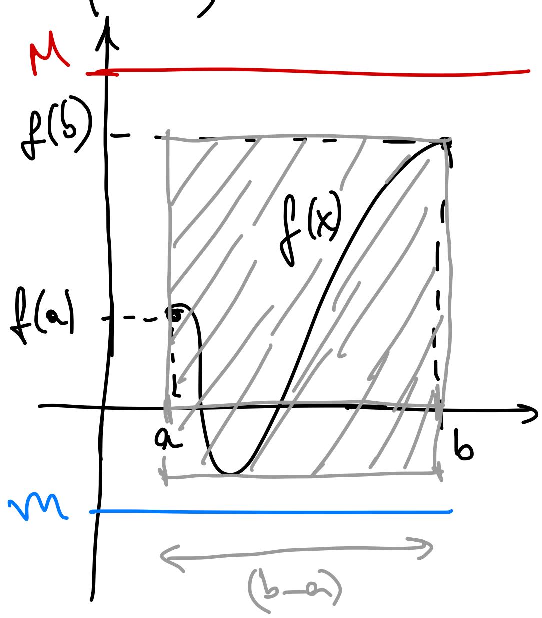
When are Riemann upper/lower integrals defined?

If $f:[a,b]\to\mathbb{R}$ is <u>bounded</u>, i.e., $\exists\, m, M\in\mathbb{R}$

$$m \le f(x) \le M \qquad \forall x\in[a,b]$$

then all the above are well-defined;

because:



$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

Since the above are bounded, $\underline{\int_a^b} f\,dx$ and $\overline{\int_a^b} f\,dx$ are well-def.

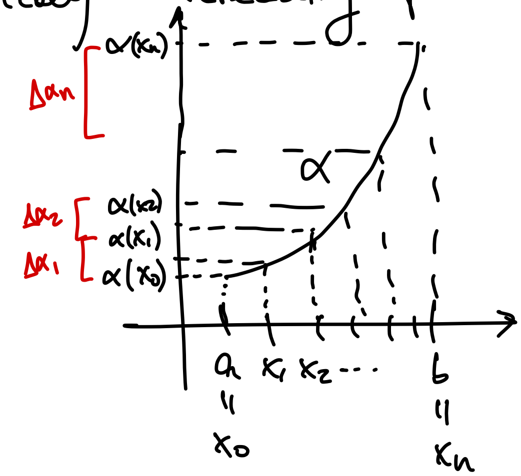
However, these quantities might not be equal to one another, i.e., f might not be $\mathcal{R}$-integrable.

Riemann-Stieltjes integral

Def: Let $\alpha : [a,b] \to \mathbb{R}$ be a monotonically increasing function

Given a partition P of $[a,b]$

$$P = \{ a = x_0 \leq x_1 \leq \dots \leq x_n = b \}$$



define

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \quad i = 1, \dots, n$$

Clearly $\Delta\alpha_i \geq 0$ since α is increasing and $x_{i-1} < x_i$.

Define upper and lower sums for a bounded function $f : [a,b] \to \mathbb{R}$ against α as follows:

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

These two quantities are the same as in the previous definition.

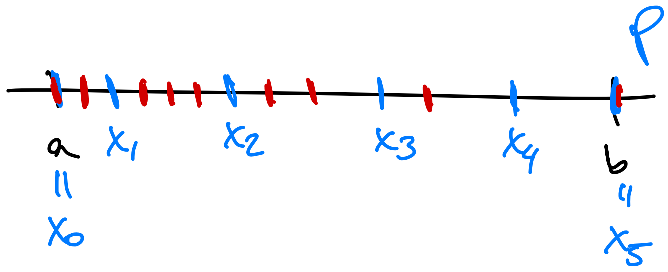
Define upper and lower integrals as before:

$$\overline{\int_a^b} f \, d\alpha = \inf_P U(P, f, \alpha), \qquad \underline{\int_a^b} f \, d\alpha = \sup_P L(P, f, \alpha)$$

Def: We say that $f : [a,b] \to \mathbb{R}$ is <u>integrable with respect</u> to $\alpha : [a,b] \to \mathbb{R}$ if $\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$, in which case we write $\int_a^b f \, d\alpha = \overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$, and $f \in \mathcal{R}(\alpha)$ ← $\mathcal{R}(\alpha)$ denotes the set of R.S.-integrable functions w.r.t. α.

Refinements of partitions

Def: We say P^* is a <u>refinement</u> of P if $P^* \supset P$.



P^* consisting of all blue and red ticks is a refinement of P, which consisted only of blue ticks.

We say P^* is the <u>common refinement</u> of P_1 and P_2 if $P^* = P_1 \cup P_2$.

Prop. If P^* is a refinement of P, then

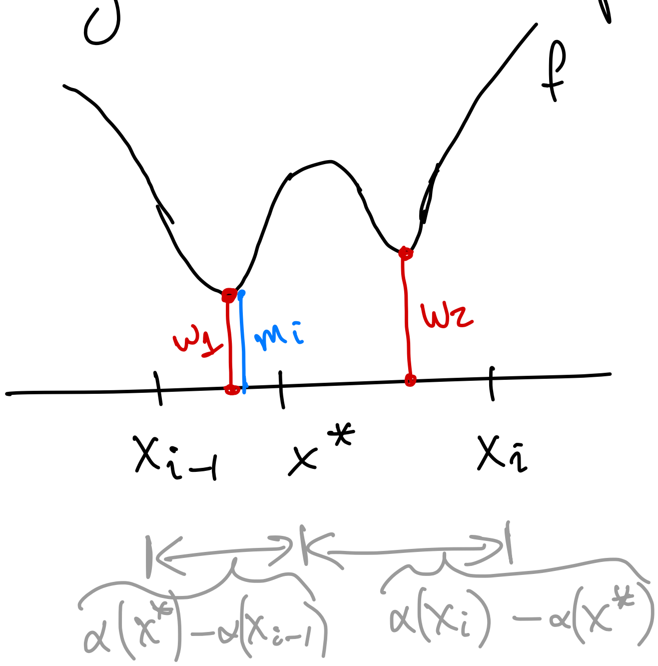
$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

$$U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

All statements will be proven with α, since they include the case $\alpha(x) = x$ which recovers the Riemann integral version from the R.S. one

$$\alpha(x) = x \Rightarrow \int f\,d\alpha = \int f\,dx$$

Pf: Suppose that $P^* = P \cup \{x^*\}$, i.e., P^* is a refinement of P by just adding one new point x^*.



Define

$$\left.\begin{aligned} w_1 &= \inf_{x \in [x_{i-1}, x^*]} f(x) \geq \underline{m_i} \\ w_2 &= \inf_{x \in [x^*, x_i]} f(x) \geq \underline{m_i} \end{aligned}\right\} \circledast$$

Hence

$$L(P^*, f, \alpha) - L(P, f, \alpha) = w_1\big(\alpha(x^*) - \alpha(x_{i-1})\big) + w_2\big(\alpha(x_i) - \alpha(x^*)\big) - m_i\big(\alpha(x_i) - \alpha(x_{i-1})\big)$$

$$= (\underbrace{w_1 - m_i}_{\substack{\geqslant 0 \\ \circledast}})\big(\underbrace{\alpha(x^*) - \alpha(x_{i-1})}_{\substack{\geqslant 0 \\ (\alpha \text{ increasing})}}\big) + (\underbrace{w_2 - m_i}_{\substack{\geqslant 0 \\ \circledast}})\big(\underbrace{\alpha(x_i) - \alpha(x^*)}_{\substack{\geqslant 0 \\ (\alpha \text{ increasing})}}\big)$$

$$\geqslant 0.$$

$$L(P^*, f, \alpha) \geqslant L(P, f, \alpha).$$

If P^* is a refinement of P through the inclusion of more points, then apply the above argument one point x^* at a time.

The proof for $U(P^*, f, \alpha) \leq U(P, f, \alpha)$ is totally analogous and is left for you as an exercise.

$\square$

__Thm.__ $\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha.$

__Pf:__ Say P_1, P_2 are partitions of $[a,b]$, and $P^* = P_1 \cup P_2$ is their common refinement. By the previous result,

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$$

So: $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

Fixing P_2 and taking a supremum over P_1, we have

$$\underline{\int_a^b} f \, d\alpha = \sup_{P_1} L(P_1, f, \alpha) \leq U(P_2, f, \alpha).$$

Taking an infimum over P_2 we conclude:

$$\underline{\int_a^b} f \, d\alpha \leq \inf_{P_2} U(P_2, f, \alpha) = \overline{\int_a^b} f \, d\alpha.$$

$\square$

Characterization of R.S.-integrability:

Thm. f is Riemann-Stieltjes integrable over $[a,b]$ w.r.t. α, i.e., $f \in \mathcal{R}(\alpha)$ $\Longleftrightarrow$ $\forall \varepsilon > 0$ $\exists P$ partition of $[a,b]$ s.t. $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

Pf: For every partition P of $[a,b]$ we know:

$$L(P, f, \alpha) \leq \underline{\int_a^b} f \, d\alpha \leq \overline{\int_a^b} f \, d\alpha \leq U(P, f, \alpha)$$

($\Longleftarrow$) If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$, then

$$0 \leq \overline{\int_a^b} f \, d\alpha - \underline{\int_a^b} f \, d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon.$$

Since this holds $\forall \varepsilon > 0$, we conclude $\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$.

($\Longrightarrow$) Conversely, $f \in \mathcal{R}(\alpha)$, let $\varepsilon > 0$. Then there exist partitions P_1 and P_2 of $[a,b]$ s.t.

$$U(P_2, f, \alpha) - \int_a^b f\, d\alpha < \frac{\varepsilon}{2}$$

$$\int_a^b f\, d\alpha - L(P_1, f, \alpha) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$ be their common refinement. From the results proven above:

$$U(P, f, \alpha) \leq U(P_2, f, \alpha) < \int_a^b f\, d\alpha + \frac{\varepsilon}{2} < L(P_1, f, \alpha) + \varepsilon$$
$$\leq L(P, f, \alpha) + \varepsilon$$

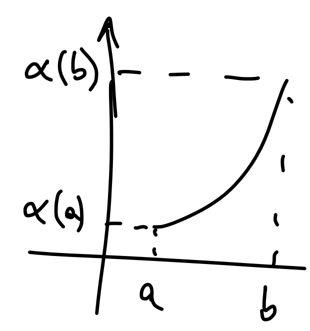
Altogether, we have

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

$\square$

<u>Thm</u>: If $f : [a,b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

<u>Pf</u>: Given $\varepsilon > 0$, choose $\eta > 0$ s.t.



$$(\alpha(b) - \alpha(a)) \cdot \eta < \varepsilon.$$

Since $f : [a,b] \to \mathbb{R}$ is cont. and $[a,b]$ is compact, we know that f is uniformly continuous on $[a,b]$. (Video 5 of Lecture 15). Thus $\exists \delta > 0$ s.t.

$$|x - t| < \delta \implies |f(x) - f(t)| < \eta. \quad (*)$$

If P is a partition of $[a,b]$, such that

$$\Delta x_i = x_i - x_{i-1} < \delta \qquad \text{for all } i = 1, 2, \ldots, n$$

then ⊛ implies $\underset{\substack{\| \\ \sup f \\ [x_{i-1}, x_i]}}{M_i} - \underset{\substack{\| \\ \inf f \\ [x_{i-1}, x_i]}}{m_i} \leq \eta$; for all $i = 1, \ldots, n$

Thus
$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} \underbrace{(M_i - m_i)}_{\leq \eta} \underbrace{\Delta\alpha_i}_{= \alpha(x_i) - \alpha(x_{i-1})}$$
$$\leq \eta \underbrace{\sum_{i=1}^{n} \Delta\alpha_i}_{\substack{\| \\ \alpha(b) - \alpha(a)}} = \eta\left(\alpha(b) - \alpha(a)\right) < \varepsilon.$$

Thus, by the Theorem above, $f \in \mathcal{R}(\alpha)$ on $[a,b]$. $\square$

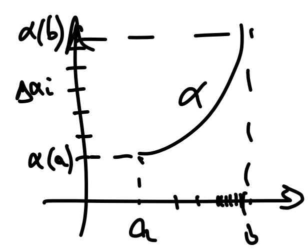
<u>Thm</u>. If $f : [a,b] \to \mathbb{R}$ is monotonic and $\alpha : [a,b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}(\alpha)$.

<u>Pf</u>: Let $\varepsilon > 0$ be given. For any $n \in \mathbb{N}$ we can choose a partition P of $[a,b]$ such that

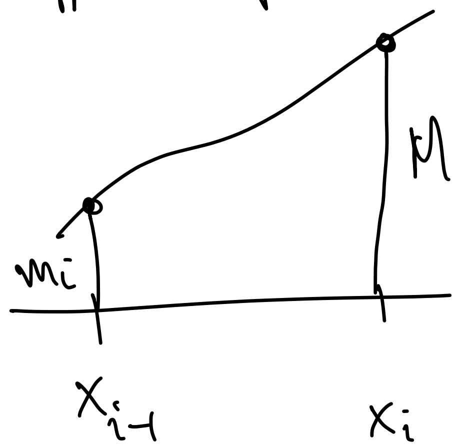
"n equal parts, as measured w.r.t. α"

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b) - \alpha(a)}{n}$$

Note this is possible by the Intermediate Value Theorem.



Suppose f is increasing (decreasing case is analogous), so:



$$M_i = f(x_i) \qquad i = 1, \ldots, n.$$

$$m_i = f(x_{i-1})$$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\, \Delta\alpha_i \qquad \Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \big(f(b) - f(a) \big) < \varepsilon$$

if n is chosen sufficiently large. By the Integrability Criterion proven above, we have that $f \in \mathcal{R}(\alpha)$. $\square$

we're always assuming: f is bounded, throughout this lecture!

Corollary: If $f : [a, b] \to \mathbb{R}$ is monotonic, then f is Riemann-integrable, i.e., $f \in \mathcal{R}$.