"Mean Value Thm" for Derivatives:

$\exists f(x)$ that are differentiable but $f'(x)$ could have some discontinuities ...

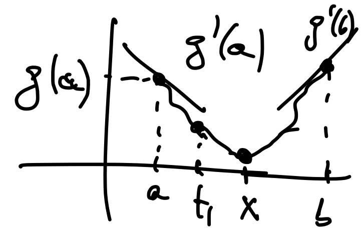
e.g., $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Thm. Suppose $f:[a,b] \to \mathbb{R}$ is differentiable and

$$f'(a) < \lambda < f'(b)$$

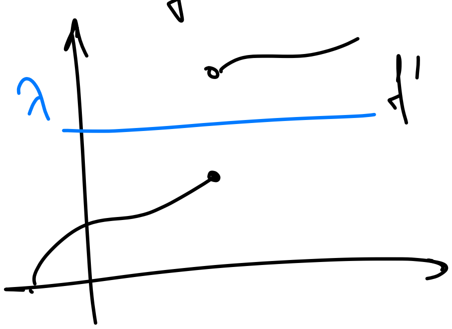
Then $\exists x \in (a,b)$ s.t. $f'(x) = \lambda$.

Pf: Let $g(t) = f(t) - \lambda t$. Then $g'(t) = f'(t) - \lambda$ so $g'(a) < 0$, hence $g(t_1) < g(a)$ for some $t_1 \in (a,b)$.



Also $g'(b) > 0$, hence $g(t_2) < g(b)$ for some $t_2 \in (a,b)$. Thus g attains its minimum at some $x \in (a,b)$. Since g is differentiable, $g'(x) = 0$, i.e. $f'(x) = \lambda$. $\square$

Cor: If $f:[a,b] \to \mathbb{R}$ is differentiable, then f' does not have discontinuities of first kind ("jumps")



This would contradict the above Thm

⚠ f' may, however, have discontinuities of second kind

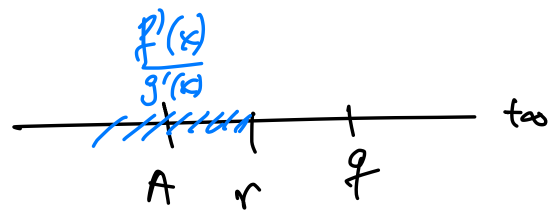
L'Hospital's Rule ← Actually obtained by Bernoulli

Thm. Suppose $f, g : (a,b) \to \mathbb{R}$ are differentiable, $g'(x) \neq 0$ for all $x \in (a,b)$, and $-\infty \le a < b \le +\infty$. Suppose

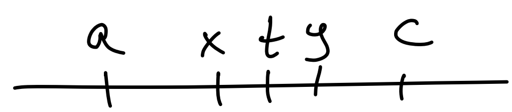
$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \qquad (-\infty \le A \le +\infty)$$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = A$.

Pf: Suppose $-\infty \le A < +\infty$. Choose $q \in \mathbb{R}$ s.t. $A < q$; then choose $r \in \mathbb{R}$ s.t. $A < r < q$.



Since $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$, $\exists c \in (a,b)$ s.t. $\frac{f'(x)}{g'(x)} < r$ if $x \in (a,c)$.



If $a < x < y < c$, then by the "Generalized" Mean Value Thm (Video 6 of Lecture 17), $\exists t \in (x,y)$ s.t.

$$\big(f(x) - f(y)\big) g'(t) = \big(g(x) - g(y)\big) f'(t).$$

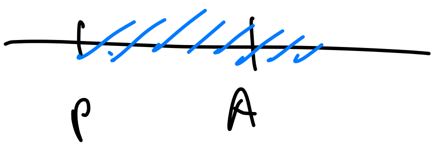
i.e.

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r \quad \text{b/c } t \in (a,c).$$

Since $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, letting $x \to a$ in the above gives:

$$\frac{f(y)}{g(y)} \leq r < q \qquad (a < y < c)$$

Analogously, if $-\infty < A \leq +\infty$ and p is such that $p < A$, then $\exists c' \in \mathbb{R}$ s.t.



$$p < \frac{f(x)}{g(x)} \qquad (a < x < c')$$

Therefore $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = A$.

Derivatives of higher order.

$$f \rightsquigarrow f' \rightsquigarrow f'' \rightsquigarrow f''' \rightsquigarrow \cdots \rightsquigarrow f^{(n)} \rightsquigarrow \cdots$$

Since the derivative $f'(x)$ of a differentiable function $f(x)$ might not be differentiable at all points, the domains of successive derivatives might be smaller than its predecessors:

$$\mathrm{Dom}\, f \supset \underbrace{\mathrm{Dom}\, f'}_{\text{where } f \text{ is differentiable}} \supset \underbrace{\mathrm{Dom}\, f''}_{\text{where } f' \text{ is differentiable}} \supset \cdots$$

Taylor's Theorem

Suppose $f(x)$ is differentiable n times on $[a,b]$.

Fix $x_0 \in (a,b)$, then

$$f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is a polynomial of degree n in x. (x_0 is fixed!)

It has the same derivatives as $f(x)$ at $x=0$.

Thm. Suppose $f : [a,b] \to \mathbb{R}$ is differentiable n times on (a,b) and $f^{(n-1)}$ is continuous on $[a,b]$. Let $\alpha < \beta$ be different points in $[a,b]$ and

"$\alpha = x_0$"

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

(polynomial of degree $(n-1)$ in the variable t)

Then there exists $x \in (\alpha, \beta)$ s.t.

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

Proof. Let $M \in \mathbb{R}$ be s.t. $\left(M = \frac{f(\beta) - P(\beta)}{(\beta-\alpha)^n}\right)$

$$f(\beta) = P(\beta) + M\underbrace{(\beta-\alpha)^n}_{\neq 0}$$

We want to show $M = \dfrac{f^{(n)}(x)}{n!}$ for some $x \in (\alpha, \beta)$.

Define $g(t) = f(t) - P(t) - M(t-\alpha)^n$, $a \le t \le b$.

Then $g^{(n)}(t) = f^{(n)}(t) - \underbrace{P^{(n)}(t)}_{=0} - M \cdot n!$

$\leftarrow$ degree $P = n-1$.

$$g^{(n)}(t) = f^{(n)}(t) - n!\, M = 0 \iff M = \frac{f^{(n)}(t)}{n!}$$

Thus, we want to show that $\exists x \in (\alpha, \beta)$ s.t. $g^{(n)}(x) = 0$.

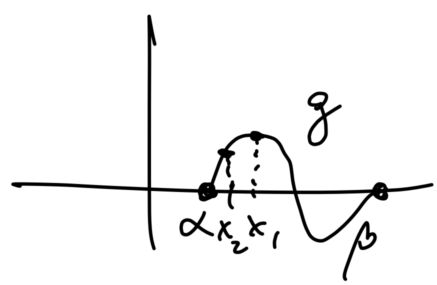
Since $\boxed{P^{(k)}(\alpha) = f^{(k)}(\alpha)}$ $\leftarrow$ Previous video: Taylor poly has exactly the same first $(n-1)$ derivatives as $f(x)$ at $x = \alpha$.

for $k = 0, 1, 2, \ldots, n-1$, we have:

$$g(\alpha) = g'(\alpha) = g''(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0.$$

Given our choice of M, we have that $g(\beta) = 0$

By the Mean Value Thm, $\exists x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = 0$. Since $g'(\alpha) = 0$ we conclude that $g''(x_2) = 0$ for some $x_2 \in (\alpha, x_1)$, again by the Mean Value Thm.



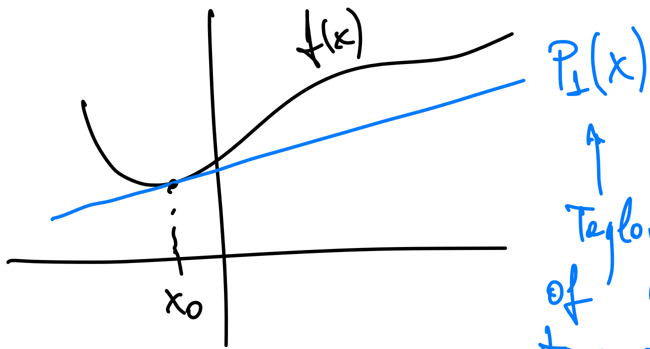
Proceeding in the same way, after n steps, we find $x_1, x_2, x_3, \dots x_n$, $\quad x_k \in (\alpha, x_{k-1})$

and $g^{(k)}(x_k) = 0$; $\quad k = 2, 3, \dots, n$. For $k = n$,

$$g^{(n)}(x_n) = 0, \qquad x_n \in (\alpha, x_{n-1}) \subset (\alpha, \beta).$$

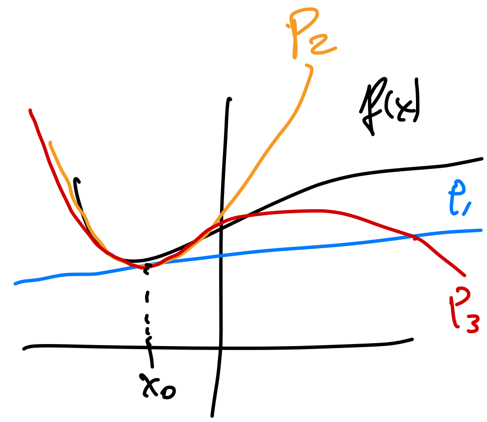
This gives our $x = x_n$ s.t. $g^{(n)}(x) = 0$. $\square$

Upshot: Any function $f: [a,b] \to \mathbb{R}$ which is differentiable n times can be approximated by a polynomial ("Taylor polynomial") of degree n.



Taylor poly of degree 1 is tangent line to $f(x)$ at $x = x_0$ ("best linear approx.")

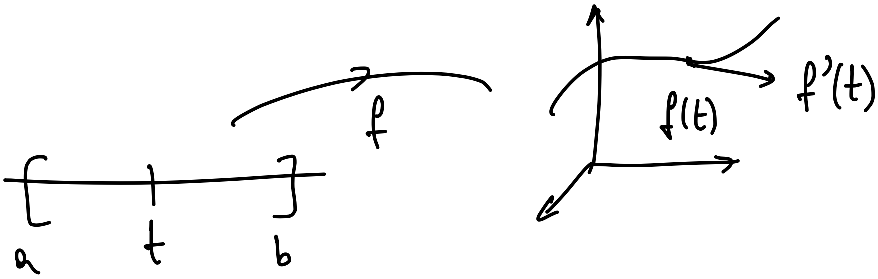
$P_1(x) = f(x_0) + f'(x_0)(x - x_0)$



Taylor polynomials $P_k(x)$ are the best k^{th} degree approx. of $f(x)$ at $x = x_0$

Derivatives of Vector-valued functions:

$f : [a,b] \to \mathbb{R}^k$



<u>Def</u>: We say $f : [a,b] \to \mathbb{R}^k$ is differentiable at $x \in [a,b]$ if

$$\lim_{t \to x} \left| \frac{f(t) - f(x)}{t - x} - \vec{v} \right| = 0$$

for some $\vec{v} \in \mathbb{R}^k$. We write $f'(x) = \vec{v}$.

- To compute derivatives, one can proceed coordinate-by-coordinate

$f : [a,b] \to \mathbb{R}^k$. $\quad f(t) = (f_1(t), f_2(t), \ldots, f_k(t))$

$$f'(t) = (f_1'(t), f_2'(t), \ldots, f_k'(t))$$

- The main result which is <u>different</u> for vector-valued functions is the Mean Value Theorem:

<u>M.V.T.</u>: $f : [a,b] \to \mathbb{R}$ cont., diff. on (a,b), then $\exists x \in (a,b)$ s.t. $f(b) - f(a) = f'(x)(b-a)$

<u>Example</u>: $f : [0, 2\pi] \to \mathbb{R}^2$, $\quad f(t) = (\cos t, \sin t) = e^{it}$

$$f'(t) = (-\sin t, \cos t) = ie^{it}$$

$$|f'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

LHS: $f(2\pi) - f(0) = (1,0) - (1,0) = (0,0)$

$$|f(2\pi) - f(0)| = \underline{\underline{0}}$$

⚠ M.V.T. does not hold in the same way.

RHS: $|f'(t)| = 1$, $|2\pi - 0| = 2\pi$, $|f'(t)(2\pi - 0)| = \underline{\underline{2\pi}}$

For $k \geq 2$, the replacement for MVT is:

<u>Theorem</u>. If $f : [a,b] \to \mathbb{R}^k$ cont., differentiable on (a,b), then $\exists x \in (a,b)$ s.t. $|f(b) - f(a)| \leq (b-a)|f'(x)|$.