MAT320 Lecture 17 10/28/2020

Derivatives.

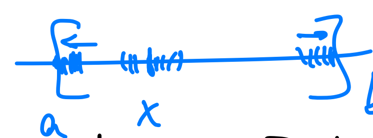
Definition. If $f:[a,b]\to\mathbb{R}$, then we define

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$$

for all $x\in[a,b]$ such that the above limit exists.

We say that f is differentiable at $x\in[a,b]$ if $f'(x)$ exists, and differentiable on $E\subset[a,b]$ if $f'(x)$ exists for all $x\in E$.

Rmk: If $f:[a,b]\to\mathbb{R}$, then f can (at most) be differentiable on $E=(a,b)$; since $f'(a)$ and $f'(b)$ cannot exist since f is only defined "to one side"



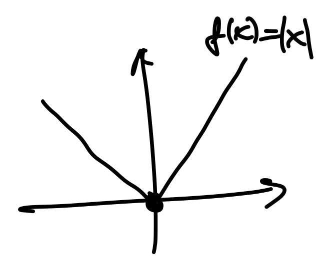
Thm. If $f:[a,b]\to\mathbb{R}$ is differentiable at $x\in[a,b]$, then f is continuous at x.

Pf: As $t\to x$,

$$f(t)-f(x) \overset{t\neq x}{=} \underbrace{\frac{f(t)-f(x)}{t-x}}_{\xrightarrow{t\to x} f'(x)} \cdot \underbrace{(t-x)}_{\to 0} \longrightarrow f'(x)\cdot 0 = 0$$

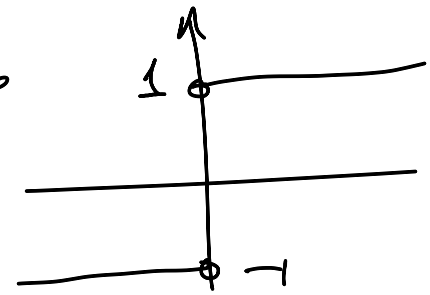
$\square$

Rmk: Converse does not hold!



$f(x) = |x|$ is continuous at all $x \in \mathbb{R}$
but it is not differentiable at $x=0$

$$\frac{f(t)-f(0)}{t-0} = \frac{|t|-0}{t} = \frac{|t|}{t} = \begin{cases} 1 & \text{if } t>0 \\ -1 & \text{if } t<0 \end{cases}$$



$$\nexists \lim_{t\to 0} \frac{f(t)-f(0)}{t-0}$$

Sum, Product, Quotient "Rules"

Thm. Suppose $f,g : [a,b] \to \mathbb{R}$ are differentiable at x.
Then $f+g$, $f \cdot g$, f/g are differentiable at x and

a) $(f+g)'(x) = f'(x) + g'(x)$

b) $(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

c) $$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$$ (If $g(x) \neq 0$)

Pf: a) $$(f+g)'(x) = \lim_{t\to x} \frac{(f+g)(t) - (f+g)(x)}{t-x} =$$

$$= \lim_{t\to x} \frac{f(t)+g(t)-f(x)-g(x)}{t-x} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x}$$

$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x} \frac{g(t)-g(x)}{t-x} = f'(x)+g'(x).$$

b) Let $h(t) = (f\cdot g)(t)$. Using basic algebra

$$\underbrace{\frac{h(t)-h(x)}{t-x}}_{h'(x)=(fg)'(x)} = \underbrace{f(t)}_{f(x)}\left(\underbrace{\frac{g(t)-g(x)}{t-x}}_{g'(x)}\right) + \underbrace{g(x)}_{g(x)}\left(\underbrace{\frac{f(t)-f(x)}{t-x}}_{f'(x)}\right) \qquad t\to x$$

c) Now $h(t) = \left(\frac{f}{g}\right)(t)$. Using basic algebra:

$$\underbrace{\frac{h(t)-h(x)}{t-x}}_{h'(x)=\left(\frac{f}{g}\right)'(x)} = \underbrace{\frac{1}{g(t)g(x)}}_{\frac{1}{g^2(x)}}\left(g(x)\underbrace{\frac{f(t)-f(x)}{t-x}}_{f'(x)} - f(x)\underbrace{\frac{g(t)-g(x)}{t-x}}_{g'(x)}\right). \qquad t\to x$$

$\square$

Remark: Applying the above rules repeatedly, one finds:

$$f(x) = x^n \rightsquigarrow f'(x) = nx^{n-1}$$

Using this, can compute the derivative of any polynomial and any rational function.

Chain Rule

$$[a \; x \; b] \xrightarrow{f} [c \;\; \underbrace{f(x)}_{\text{Im} f} \;\; d] \xrightarrow{g} g(f(x)) \in \mathbb{R}$$

Thm. Suppose $f : [a,b] \to \mathbb{R}$ is continuous, and $f'(x)$ exists at some $x \in [a,b]$. Suppose $f([a,b]) \subset [c,d]$ and $g : [c,d] \to \mathbb{R}$ is differentiable at $f(x)$. Then $h(t) = g(f(t))$ is differentiable at x and $h'(x) = g'(f(x)) \cdot f'(x)$.

Pf. Let $y = f(x)$. Since f is diff. at x, and g is diff. at $f(x)$,

$$f(t) - f(x) = (t-x)\big(f'(x) + u(t)\big) \text{ where } t \in [a,b],\ u(t) \to 0 \text{ as } t \to x$$

$$g(s) - g(y) = (s-y)\big(g'(y) + v(s)\big) \text{ where } s \in [c,d],\ v(s) \to 0 \text{ as } s \to y$$

Let $s = f(t)$, using the above:

$$h(t) - h(x) = g(\underbrace{f(t)}_{s}) - g(\underbrace{f(x)}_{y})$$

$$= \big(\underbrace{\underbrace{f(t)}_{s} - \underbrace{f(x)}_{y}}\big)\big(g'(\underbrace{f(x)}_{y}) + v(s)\big)$$

$$= (t-x)\big(f'(x) + u(t)\big)\big(g'(f(x)) + v(s)\big)$$

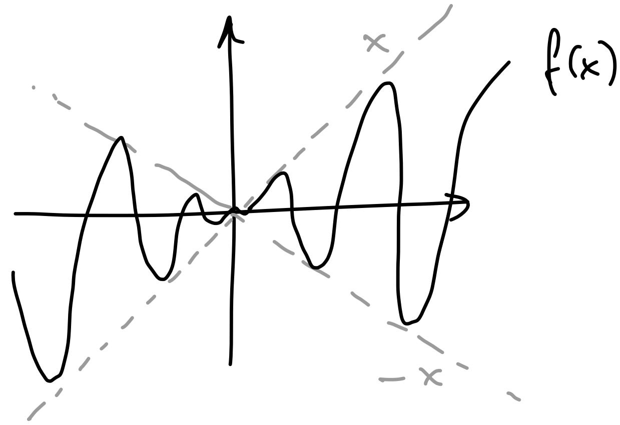
if $t \neq x$,

$$\underbrace{\frac{h(t) - h(x)}{t - x}}_{h'(x) = (g\circ f)'(x)} = \big(f'(x) + \underbrace{u(t)}_{\to 0}\big)\big(g'(f(x)) + \underbrace{v(s)}_{\to 0}\big) \longrightarrow f'(x) \cdot g'(f(x))$$

$t \to x$ $s \to y$, $v(s) \to 0$

$\square$

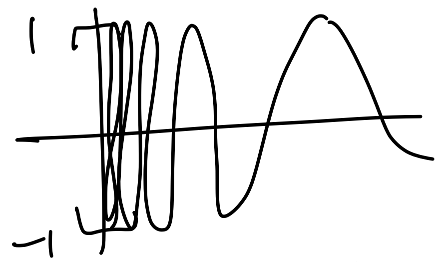
Example.



$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f'(x) = 1 \cdot \sin \frac{1}{x} + x \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$$

$$= \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}. \quad \text{if } x \neq 0.$$

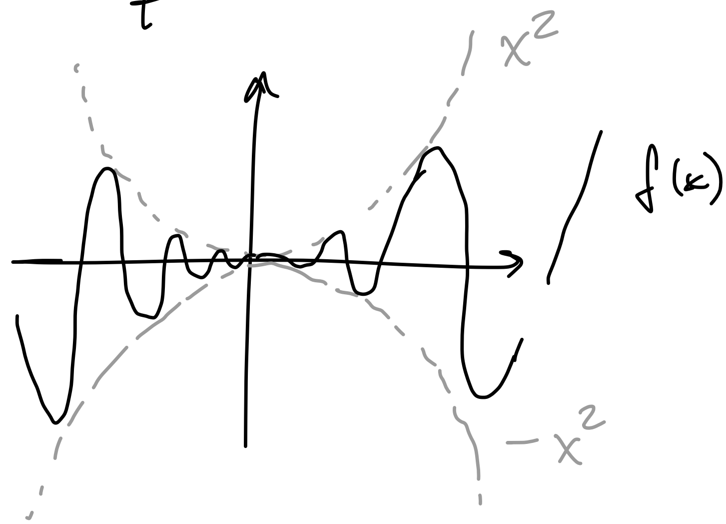


$f'(0)$ does not exist:

$$\frac{f(t) - f(0)}{t - 0} = \frac{t \sin \frac{1}{t}}{t} = \sin \frac{1}{t} \text{ has no limit as } t \to 0.$$

Now let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



$$f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \left(-\frac{1}{x^2}\right)$$

$$= 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad \text{if } x \neq 0.$$

$$\frac{f(t) - f(0)}{t - 0} = \frac{t^2 \sin \frac{1}{t}}{t} = t \sin \frac{1}{t} \text{ \underline{has} a limit as } t \to 0$$

(by Squeeze Thm).

$$f'(0) = \lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = 0.$$

So this $f(x)$ is differentiable <u>everywhere</u>.

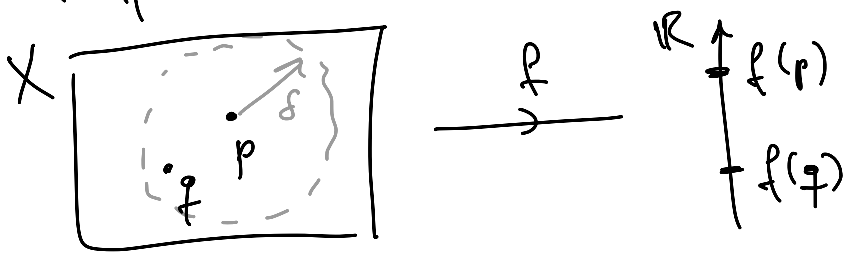
But;

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

So $f'(x)$ is not continuous at $x=0$!

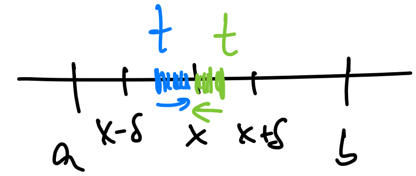
Local Maxima/Minima

<u>Def</u>: $f: X \to \mathbb{R}$ has a <u>local max</u> at $p \in X$ if $\exists \delta > 0$ s.t. if $q \in X$ and $d(p,q) < \delta$, then $f(q) \le f(p)$



<u>Thm</u>. If $f: [a,b] \to \mathbb{R}$ has a <u>local max</u> at $x \in [a,b]$ and $f'(x)$ exists, then $f'(x) = 0$.

<u>Pf</u>: Take $\delta > 0$ s.t. $[x-\delta, x+\delta] \subset [a,b]$



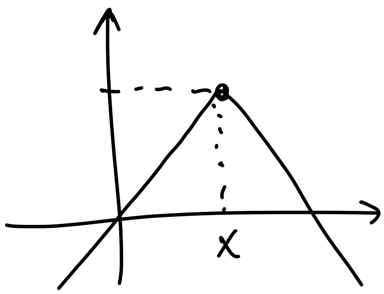
and note:

if $t \in [x-\delta, x]$, then $\dfrac{f(t)-f(x)}{t-x} \geqslant 0 \Rightarrow \lim_{t\to x_-} \dfrac{f(t)-f(x)}{t-x} \geqslant 0$

if $t \in [x, x+\delta]$, then $\frac{f(t)-f(x)}{t-x} \leq 0 \Rightarrow \lim_{t \to x_+} \frac{f(t)-f(x)}{t-x} \leq 0$

Since $f'(x)$ exists, both the above lateral limits exist and are equal to one another. Therefore both equal 0, and hence $f'(x)=0$. $\square$

<ins>Remark</ins>:



A function $f: [a,b] \to \mathbb{R}$ can have a local max. at $x \in [a,b]$ without being differentiable at x!

<ins>Mean Value Theorems</ins>

<ins>Thm</ins>. If $f, g: [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), then $\exists x \in (a,b)$ s.t.

$$\big(f(b)-f(a)\big)\, g'(x) = \big(g(b)-g(a)\big)\, f'(x)$$

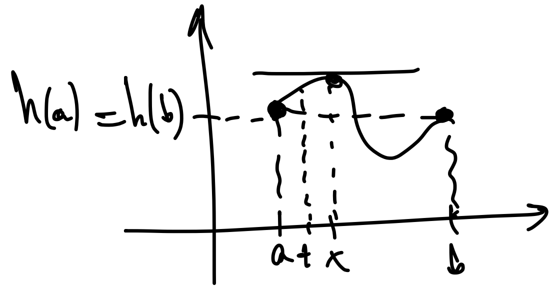
<ins>Pf</ins>: Let $h: [a,b] \to \mathbb{R}$

$$h(t) = \big(f(b)-f(a)\big)g(t) - \big(g(b)-g(a)\big)f(t)$$

By hypotheses of f, g, h is cont. on $[a,b]$ and diff. on (a,b).

Moreover $h(a) = \big(f(b)-f(a)\big)g(a) - \big(g(b)-g(a)\big)f(a) = h(b)$

If $h(t)$ is constant, then $h'(t)=0$ for all t, so let $x\in[a,b]$ be any point.



If $h(t)$ is <u>not</u> constant, then $\exists t\in(a,b)$ s.t. $h(t)>h(a)$. Since h is continuous and $[a,b]$ is compact, there exists a global max. $x\in[a,b]$ for h. By Theorem above, $h'(x)=0$, i.e. $\big(f(b)-f(a)\big)g'(x)-\big(g(b)-g(a)\big)f'(x)=0$

If, instead, $\exists t\in[a,b]$ s.t. $h(t)<h(a)$. Following the same reasoning as above, there exists a global min $x\in[a,b]$ for h, and, as before $h'(x)=0$. $\square$

<u>Corollary</u>. If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists x\in(a,b)$ s.t.

$$f(b)-f(a)=f'(x)\,(b-a)$$

$$\underbrace{f'(x)}_{m}=\underbrace{\frac{f(b)-f(a)}{b-a}}_{m}$$

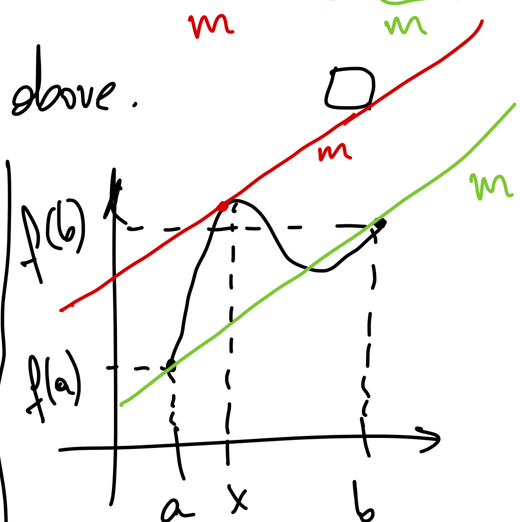
<u>Pf</u>: Take $g(x)=x$ in the Theorem above. $\square$

<u>Corollary</u>: If $f:(a,b)\to\mathbb{R}$ is differentiable,

a) $f'(x)\geq 0,\ \forall x\in(a,b)\Rightarrow f(x)$ is monoton. increasing

b) $f'(x)=0,\ \forall x\in(a,b)\Rightarrow f(x)$ is constant

c) $f'(x)\leq 0,\ \forall x\in(a,b)\Rightarrow f(x)$ is monoton. decreasing.



Pf: Take <u>any</u> $x_1, x_2 \in (a,b)$, $x_1 < x_2$ and use that

$$f(x_2) - f(x_1) = (x_2 - x_1) \cdot f'(x)$$

for some $x \in (x_1, x_2)$. $\square$