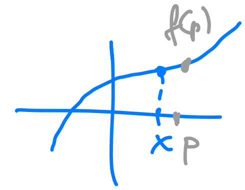
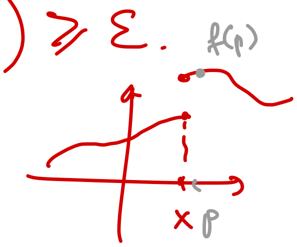
Discontinuities:

Def: A function $f: X \to Y$ is <u>discontinuous</u> at $x \in X$ if it is <u>not</u> continuous at $x \in X$.



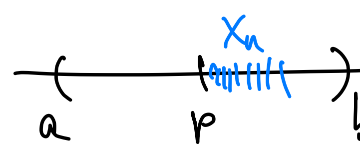
<u>Continuous at x:</u> $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
$$0 < d(x,p) < \delta \implies d(f(x), f(p)) < \varepsilon.$$

<u>Not continuous at x:</u> $\exists \varepsilon > 0 \ \forall \delta > 0$
$$0 < d(x,p) < \delta \implies d(f(x), f(p)) \geq \varepsilon.$$



Def: (Lateral limits). Let $f: (a,b) \to Y$ be a function. Then given $p \in (a,b)$,

(Right limit) $$\lim_{x \to p_+} f(x) = f(p_+) = q$$

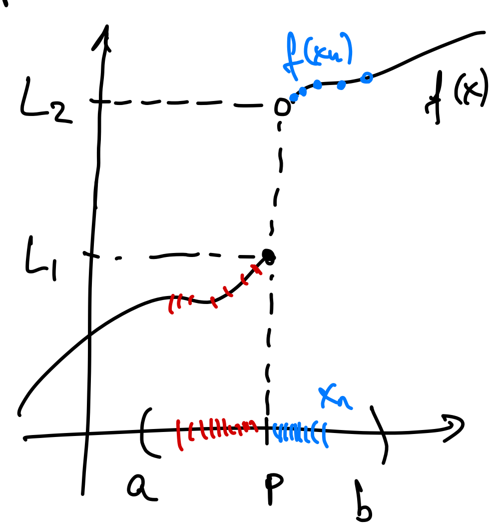


if $f(x_n) \to q$ for all sequences $\{x_n\}$ in (p,b) s.t. $x_n \to p$. Analogously for left limits:

$$\lim_{x \to p_-} f(x) = f(p_-)$$

In the picture;

$$\lim_{x \to p_+} f(x) = L_2, \quad \lim_{x \to p_-} f(x) = L_1$$



Recall: $\lim_{x\to p} f(x)$ exists if and only if both lateral limits exist and

$$\lim_{x\to p_+} f(x) = \lim_{x\to p_-} f(x).$$

Discontinuities of first and second kind

Def: We say f has a discontinuity of <u>first kind</u> at $p\in(a,b)$ if f is discontinuous at p but the lateral limits $\lim_{x\to p_+} f(x)$ and $\lim_{x\to p_-} f(x)$ exist.

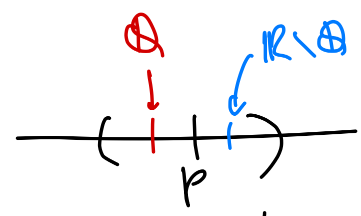
If f is discontinuous at p, and (at least one of) the lateral limits does not exist then we say the discontinuity is of <u>second kind</u>.

Examples:

1) $f:\mathbb{R}\to[0,1]$

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$$

Since $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are both <u>dense</u> in $\mathbb{R}$



this function $f(x)$ is discontinuous at all $p\in\mathbb{R}$.

Since <u>none</u> of the lateral limits exist at any $p\in\mathbb{R}$, these discontinuities are of <u>second kind</u>.

2) $f: \mathbb{R} \to \mathbb{R}$

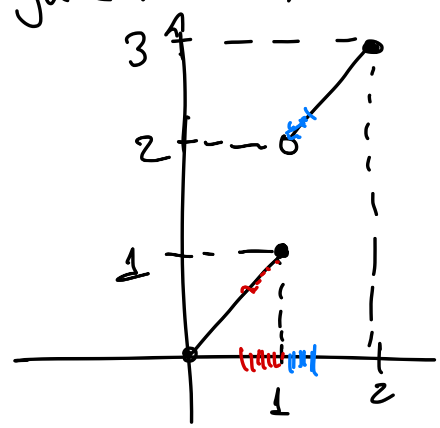
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- continuous at $x = 0$, discontinuous at all other points
- The above discontinuities are of <u>second kind</u>.

<u>Exercise</u>: Write details of these claims using <u>sequences</u>!

(Remember: $\forall p \in \mathbb{R}$, $\exists \{x_n\}, \{y_n\}$ sequences w/ $x_n \to p$, $y_n \to p$ and $x_n \in \mathbb{Q}, \forall n \in \mathbb{N}$, $y_n \in \mathbb{R} \setminus \mathbb{Q}$, $\forall n \in \mathbb{N}$)

3) $$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x+1 & \text{if } x \in (1,2] \end{cases}$$



This function is continuous on $[0,2] \setminus \{1\}$ and discontinuous at $p = 1$. This discontinuity is of first kind:

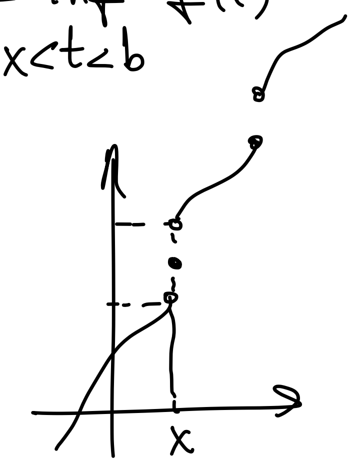
$$\lim_{x \to 1_+} f(x) = 2, \qquad \lim_{x \to 1_-} f(x) = 1.$$

<u>Definition</u>: We say $f: (a,b) \to \mathbb{R}$ is <u>monotonically increasing</u> if $a < x < y < b \Rightarrow f(x) \le f(y)$; similarly, it is <u>monotonically decreasing</u> if $a < x < y < b \Rightarrow f(x) \ge f(y)$.

A function f is <u>monotonic</u> if it is either of the above.

Thm. Let $f : (a,b) \to \mathbb{R}$ be monotonically increasing. Then its lateral limits exist at all $x \in (a,b)$, and

$$\sup_{a<t<x} f(t) = \lim_{t \to x_-} f(t) \le f(x) \le \lim_{t \to x_+} f(t) = \inf_{x<t<b} f(t)$$



Moreover, if $a < x < y < b$, then

$$\lim_{t \to x_+} f(t) \le \lim_{t \to y_-} f(t)$$

(An analogous statement holds for monotonically decreasing functions; e.g., replace $f(x)$ by $-f(x)$ in statement)

Corollary: Monotonic functions do not have discontinuities of the second kind.

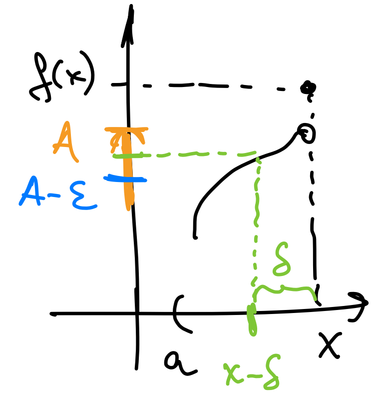
Proof. Since f is monotonic, the set

$$\{ f(t) \, ; \, a < t < x \}$$

is bounded from above, e.g., by $f(x)$. Therefore, it has a least upper bound:

$$A := \sup \{ f(t) : a < t < x \} = \sup_{a<t<x} f(t)$$

Claim: $\lim_{t \to x_-} f(t) = A$



Given $\varepsilon > 0$, since A is the least upper bound, $\exists \delta > 0$ s.t. $a < x - \delta < x$ and $A - \varepsilon < f(x-\delta) \leq A$. Since f is monotonic, increasing

$$f(x-\delta) \leq f(t) \leq A \quad \text{for all} \quad x - \delta < t < x$$

Combining the above:

$$A - \varepsilon < f(x-\delta) \leq f(t) \leq A$$
$$-\varepsilon < f(t) - A \leq 0 < \varepsilon$$
$$|f(t) - A| < \varepsilon.$$

$$x - \delta < t < x \implies |f(t) - A| < \varepsilon$$

The above means precisely that $\lim_{t \to x_-} f(t) = A$.

Similarly, one uses the exact same procedure to show $\lim_{t \to x_+} f(t) = \inf_{x<t<b} f(t)$. Finally, given $a < x < y < b$, from the above

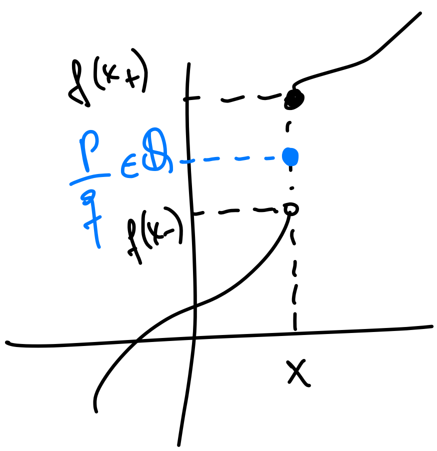
$$\lim_{t \to x_+} f(t) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t)$$
$$\geq$$

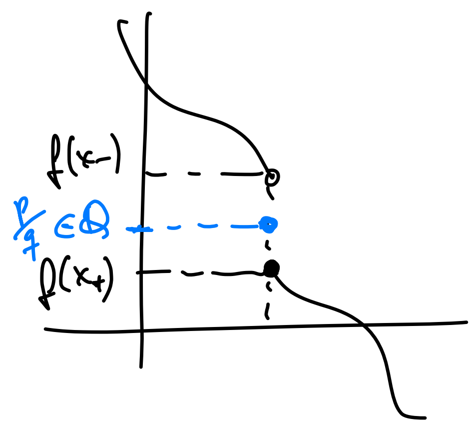
Analogously, $\lim_{t \to y_-} f(t) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t)$

proving the last part of the statement. $\square$

Corollary: The set of discontinuities of a monotonic function is countable.

Pf: Since a monotonic function f only has discontinuities of first kind, we can place a rational number between the lateral limits at every discontinuity





$$f(x_\pm) = \lim_{t \to x_\pm} f(t)$$

Since f is monotonic, these rational numbers are all distinct. Therefore, the set of discontinuities of f is in 1-1 correspondence with a subset of $\mathbb{Q}$; hence is countable. $\square$

Remark: Despite being countable, the discontinuities of monotonic functions might accumulate.

Given any countable set $E \subset \mathbb{R}$ (e.g., $E = \mathbb{Q}$), one can build a monotonic increasing function $f: \mathbb{R} \to \mathbb{R}$ that is discontinuous at all points of E but continuous everywhere else:

Say $E = \{x_n : n \in \mathbb{N}\} = \{x_1, x_2, x_3, \dots\}$

Let $\{c_n\}$ be a seq. of positive real numbers s.t. $\sum_{n=1}^{+\infty} c_n < \infty$; e.g., $c_n = \frac{1}{n^2}$. Define

$$f(x) = \sum_{\{n : x_n < x\}} c_n$$

Clearly $f(x)$ is monot. increasing and discont. at every x_n:

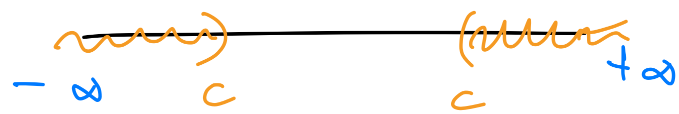
$$\lim_{t \to x_n +} f(t) - \lim_{t \to x_n -} f(t) = c_n$$

and cont. (even locally constant) at every $x \notin E$.

Infinite limits & Limits at infinity

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ extended real line

Def: $\forall c \in \mathbb{R}$, the unbounded interval $(c, +\infty)$ is a neighborhood of $+\infty$, and $(-\infty, c)$ is a neighborhood of $-\infty$.



extends our earlier definition of limits to include $\{\pm\infty\}$.

Def: Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be a function. We say

$$\lim_{t \to x} f(t) = A$$

where $A, x \in \overline{\mathbb{R}}$, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E \neq \emptyset$ and $f(t) \in U$ whenever $t \in (V \cap E) \setminus \{x\}$.

With the above definition, one can rigorously deal with limits at infinity $(x = \pm\infty)$ and/or infinite limits $(A = \pm\infty)$. It also, of course, matches our earlier def. for real numbers

Examples: $\displaystyle\lim_{t \to +\infty} \frac{t^2}{1+t^2} = 1$, $\displaystyle\lim_{x \to -\infty} e^x = 0$, ...

Thm: Let $f, g : E \subset \mathbb{R} \to \mathbb{R}$ and suppose

$$\lim_{t \to x} f(t) = A, \qquad \lim_{t \to x} g(t) = B.$$

where $x, A, B \in \overline{\mathbb{R}}$. Then

(i) $\lim\limits_{t \to x} f(t) = A'$ then $A' = A$ (uniqueness)

(ii) $\lim\limits_{t \to x} (f+g)(t) = A + B$

(iii) $\lim\limits_{t \to x} (f \cdot g)(t) = A \cdot B$

(iv) $\lim\limits_{t \to x} \left(\frac{f}{g}\right)(t) = \frac{A}{B}$

provided the right-hand side of the above is well-defined.

Recall:

$$\left.\begin{array}{c} +\infty - \infty \\ -\infty + \infty \\ 0 \cdot \pm\infty \\ \dfrac{\infty}{\infty} \\ \dfrac{A}{0} \end{array}\right\} \text{ are } \underline{\text{not}} \text{ well-defined.}$$

The following are well-def.

$+\infty + \infty = +\infty$

$-\infty - \infty = -\infty$

$c \cdot +\infty = +\infty$ if $c > 0$

...