MAT 320 Lecture 12 10/7/2020

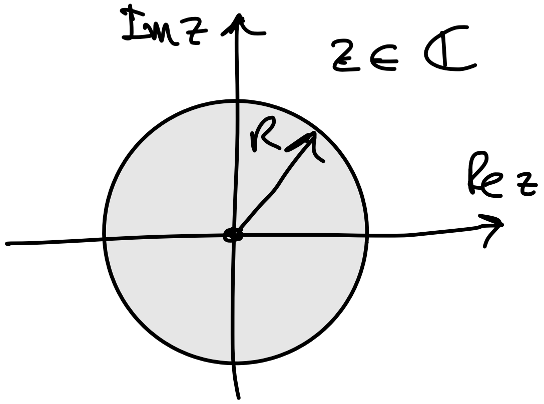
Power Series

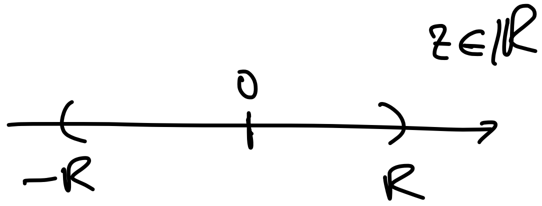
Given a sequence $\{a_n\}$ of (real or complex) numbers, let

Power series

$$\sum_{n=0}^{+\infty} \underset{\text{coeff.}}{a_n} \underset{\text{variable}}{z}^{n}$$

(function of z is defined if z is within distance R from 0)





Thm: The radius of convergence R for the power series $\sum_{n=0}^{+\infty} a_n z^n$ is given by:

$$\frac{1}{R} = \underbrace{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

Quantity that appears in the Root test.

Pf: Let $c_n := a_n z^n$ and apply Root test to

$$\sum_{n=0}^{+\infty} c_n = \sum_{n=0}^{+\infty} a_n z^n .$$

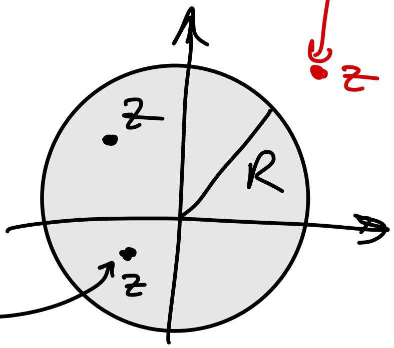
$$\begin{aligned}
\limsup_{n\to\infty} \sqrt[n]{|c_n|} &= \limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} \quad = \sqrt[n]{|a_n|\,|z|^n} \\
&= \limsup_{n\to\infty} \underbrace{|z|}_{\text{does not depend on } n} \sqrt[n]{|a_n|} \\
&= |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \frac{|z|}{R} .
\end{aligned}$$

Then $\sum_{n=0}^{+\infty} c_n = \sum_{n=0}^{+\infty} a_n z^n$ converges if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} < 1$,

i.e., if $\frac{|z|}{R} < 1$, i.e. if $|z| < R$.

Series diverges if z is here

(series converges if z is here)



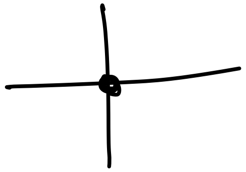
Similarly, $\sum_{n=0}^{+\infty} c_n = \sum_{n=0}^{+\infty} a_n z^n$ diverges if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} > 1$

i.e., if $|z| > R$. $\square$

Remark: If $|z| = R$, then power series might converge or diverge
(Root test is inconclusive)

Examples:

a) $\sum_{n=0}^{+\infty} n^n z^n$ $\quad a_n = n^n \quad$ $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} n = +\infty$



$R = 0$ radius of convergence.

(cf. n^{th} term test if $|z| > 0$ ...)

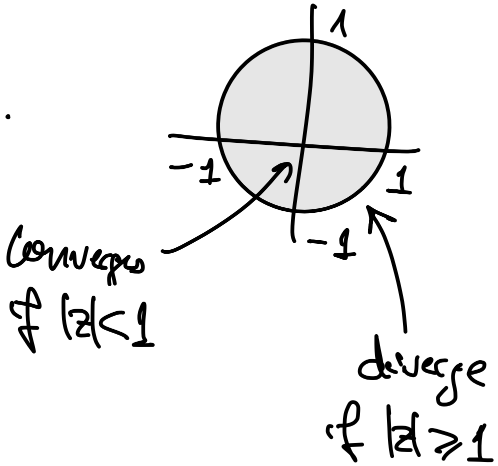
b) $\sum_{n=0}^{+\infty} \frac{z^n}{n!}$ Easier to use ratio test

$a_n = \frac{1}{n!}$

$$\frac{1}{R} = \limsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \limsup_{n\to\infty} \left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \lim_{n\to\infty} \frac{|z| \cdot \cancel{n!}}{\cancel{(n+1)!}} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0.$$

$\Rightarrow R = +\infty$. Series converges for all $z \in \mathbb{C}$

c) $\sum_{n=0}^{+\infty} z^n$ $\quad \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1. \Rightarrow R = 1.$

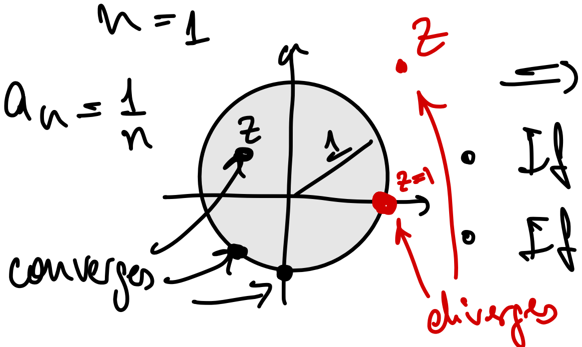


$a_n = 1$ $\quad$ If $|z|=1$, then $|z^n| = 1^n = 1 \not\to 0$

so series diverges for all such z.

d) $\sum_{n=1}^{+\infty} \frac{z^n}{n}$ $\quad \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$

$a_n = \frac{1}{n}$



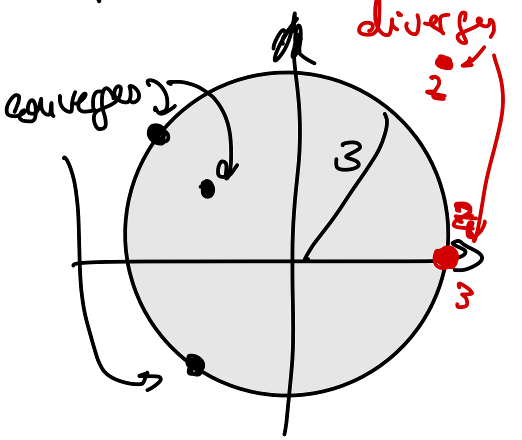
$\Rightarrow R = 1$

- If $z=1$, then this is the harmonic series $\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$. (diverges)
- If $|z|=1$ but $z=1$ then it converges

e) $\sum_{n=1}^{+\infty} \frac{z^n}{n3^n} = \sum_{n=1}^{+\infty} \frac{w^n}{n}$ converges iff $|w| \le 1$ and $w \neq 1$, since $w = \frac{z}{3}$, we have convergence iff $|z| \le 3$, $z \neq 3$.

$R=3$

f) $\sum_{n=1}^{+\infty} \frac{z^n}{n^2}$ $\quad \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \limsup_{n\to\infty} \frac{1}{(\sqrt[n]{n})^2} = 1.$

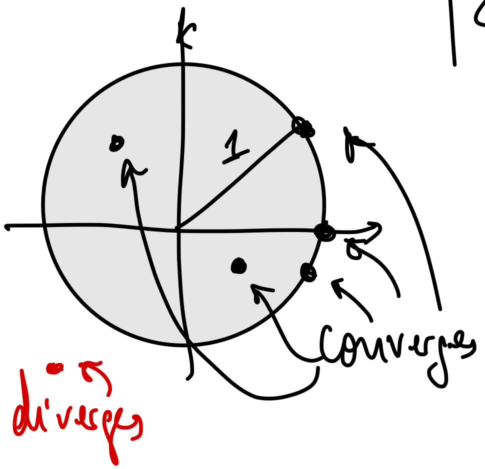


$\Rightarrow R = 1. \Rightarrow$ converges for all $|z| < 1$.

We have convergence for all $|z| \le 1$.

p - Series $p=2$

$|z|=1$: $$\left|\sum_{n=1}^{+\infty} \frac{z^n}{n^2}\right| \le \sum_{n=1}^{+\infty} \left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{+\infty} \frac{|z|^n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$$



Summation by Parts

Lemma: Given $\{a_n\}$ and $\{b_n\}$ sequences, let for $n \in \mathbb{N}$

$$A_n = \sum_{k=0}^{n} a_k \leftarrow$$ (Partial sum of $\sum a_n$)

and $A_{-1}=0$. Then if $0 \le p \le q$,

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Pf: $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} \underbrace{(A_n - A_{n-1})}_{a_n} b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \; . \; \square$$

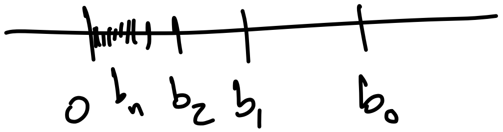
shift index by 1.

Thm: Suppose

(i) Partial sums A_n of $\sum a_n$ form a bounded sequence

(ii) $b_0 \geq b_1 \geq b_2 \geq \ldots$ (b_n is monotonically decreasing)

(iii) $\lim\limits_{n \to \infty} b_n = 0$.



Then $\sum\limits_{n=0}^{+\infty} a_n b_n$ converges.

Pf: Let $M \in \mathbb{R}$ s.t. $|A_n| \leq M$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$ let $N \in \mathbb{N}$ be s.t. $b_N \leq \frac{\varepsilon}{2M}$. If $N \leq p \leq q$, then:

$$\left|\sum_{n=p}^{q} a_n b_n\right| \overset{\substack{\text{Lemma}\\ \downarrow}}{=} \left|\sum_{n=p}^{q-1} \underbrace{A_n}_{|\cdot|\le M}(b_n-b_{n+1}) + \underbrace{A_q}_{|\cdot|\le M} b_q - \underbrace{A_{p-1}}_{|\cdot|\le M} b_p\right|$$

$$\le M \underbrace{\left|\sum_{n=p}^{q}\underbrace{(b_n-b_{n+1})}_{\nwarrow \text{ telescopic sum}} + b_q - b_p\right|}_{2b_p}$$

$$(\underline{b_p} - \cancel{b_{p+1}}) + (\cancel{b_{p+1}} - \cancel{b_{p+2}}) + (\cancel{b_{p+2}} - b_{p+3}) + \cdots \nearrow$$

$$= 2Mb_p$$

$$\xrightarrow{p \geq N} \le 2Mb_N \le \varepsilon.$$

This implies convergence of $\sum a_n b_n$ by Cauchy Criterion (Lecture 10 Video 3).

□

As a consequence:

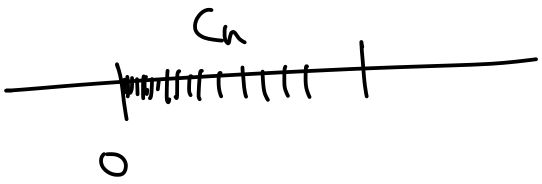
$$c_0 - c_1 + c_2 - c_3 + c_4 - \dots$$

$$\|$$

<u>Alternating Series Test</u>: The alternating series $\sum_{n=0}^{+\infty} (-1)^n c_n$ <u>converges</u> if

(i) $c_n \geq 0$ is monotonically decreasing sequence

$$c_0 \geq c_1 \geq c_2 \geq c_3 \geq \cdots$$



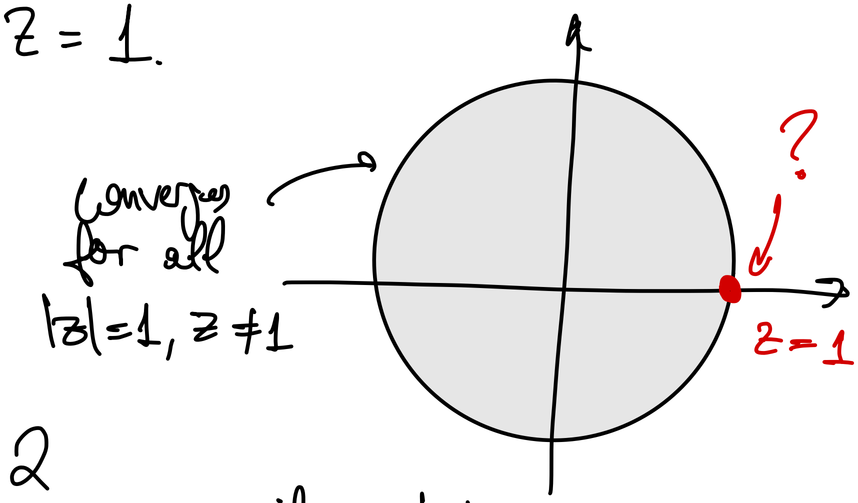
(ii) $\lim_{n \to \infty} c_n = 0$.

<u>Pf</u>: Apply previous thm with $a_n = (-1)^n \longrightarrow |A_n| \leq 1$

$b_n = c_n$.

Another application:

Cor: If radius of convergence of $\sum c_n z^n$ is 1 and $c_0 \geq c_1 \geq c_2 \geq \dots$, $\lim_{n\to\infty} c_n = 0$, then $\sum c_n z^n$ converges for all $|z|=1$, except possibly $z=1$.



Pf: Put $a_n = z^n$, $b_n = c_n$.

$$|A_n| = \left| \sum_{m=0}^{n} z^m \right| = \left| \frac{1-z^{n+1}}{1-z} \right| \leq \frac{2}{|1-z|} \quad \text{if } |z|=1, \ z \neq 1.$$

Geometric sum.

$\square$

(This confirms what we claimed in Example d) above!)

Absolute convergence and Conditional Convergence

Def. $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Prop: If $\sum a_n$ converges absolutely, then it converges.

Pf:

$$\left|\sum a_n\right| \overset{\text{triangle ineq.}}{\le} \underbrace{\sum |a_n|}_{\text{converges.}} . \qquad \square$$

Rmk: The converse does not hold in general...

Def: $\sum a_n$ converges conditionally if it converges, but not absolutely.

E.g.: Alternating harmonic series converges conditionally.

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \overset{\text{by Alt. Series Test.}}{<} +\infty \quad \text{but} \quad \sum_{n=1}^{+\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{+\infty} \frac{1}{n} \overset{\text{diverges.}}{=} +\infty$$

Addition and Multiplication of Series

Let $\sum a_n$, $\sum b_n$ be series

Prop: If $\sum a_n = A$ and $\sum b_n = B$, then

(i) $\sum a_n + b_n = A + B$

(ii) $\sum c . a_n = c . A$ for any $c \in \mathbb{R}$.

S_n = Partial sum for $\sum a_n$

t_n = partial sum for $\sum t_n$

Pf: Follows from $\lim\limits_{n\to\infty} S_n + t_n = \lim\limits_{n\to\infty} s_n + \lim\limits_{n\to\infty} t_n$

$$\lim_{n\to\infty} c . s_n = c . \lim_{n\to\infty} s_n .$$

□

But product of 2 series is a bit more delicate...

$$\left(\sum_{n=0}^{+\infty} a_n z^n\right)\left(\sum_{n=0}^{+\infty} b_n z^n\right) = \left(a_0 + a_1 z + a_2 z^2 + \dots\right)\left(b_0 + b_1 z + b_2 z^2 + \dots\right)$$

(z^2)

① ⓩ

$$= \underline{a_0 b_0} + \underline{(a_0 b_1 + a_1 b_0) z}$$

$$+ \underline{(a_0 b_2 + a_1 b_1 + a_2 b_0)} z^2 + \dots$$

$$= \sum_{n=0}^{+\infty} c_n z^n$$

Let $\quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$ ← natural candidate to be the product $\left(\sum a_n\right)\left(\sum b_n\right)$.

⚠ It might not converge, <u>even if</u> $\sum a_n, \sum b_n$ do.

A tricky example: Product of convergent series may diverge

$$A = \sum_{n=0}^{+\infty} \boxed{\frac{(-1)^n}{\sqrt{n+1}}}^{\,=a_n} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots < +\infty$$

converges!

Alternating Series

$$\frac{1}{\sqrt{n+1}} \longrightarrow 0.$$

Let's multiply this series by itself

$$c_n = \sum_{k=0}^{n} a_k \, a_{n-k} = \sum_{k=0}^{n} \underbrace{\frac{(-1)^k}{\sqrt{k+1}}}_{a_k} \cdot \underbrace{\frac{(-1)^{n-k}}{\sqrt{n-k+1}}}_{a_{n-k}}$$

$$= \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{(k+1)(n-k+1)}}$$

No!

$$\underset{\text{finite}}{A^2} = \underset{\text{diverges!}}{\sum_{n=0}^{\infty} c_n} \; ?$$

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \underbrace{\left(\frac{n}{2}-k\right)^2}_{\geqslant 0} \leq \left(\frac{n}{2}+1\right)^2$$

$$\frac{1}{\sqrt{(k+1)(n-k+1)}} \geqslant \frac{1}{\frac{n}{2}+1} = \frac{2}{n+2}.$$

does not depend on k

$$|c_n| \geqslant \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}} \geqslant \sum_{k=0}^{n} \frac{2}{n+1} = \frac{2(n+1)}{n+2}$$

$$\lim_{n\to\infty} |c_n| \geqslant \lim_{n\to\infty} \frac{2n+2}{n+2} = 2 \qquad \underline{c_n \not\to 0}$$

By the "n^{th} term test", the series $\underline{\sum c_n \text{ diverges..}}$

□

Sufficient condition for convergence of products:

Thm. Suppose $\sum_{n=0}^{+\infty} a_n$ converges absolutely, $\sum_{n=0}^{+\infty} a_n = A$

and $\sum_{n=0}^{+\infty} b_n = B$ converges. Then, letting

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$, we have $\sum_{n=0}^{+\infty} c_n = A \cdot B$.

Pf: Consider the partial sums:

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k, \quad \beta_n := B_n - B.$$

Note since $B_n \to B$, also $\beta_n \to 0$.

Then;

$$C_n = \underbrace{a_0 b_0}_{c_0} + \underbrace{(a_0 b_1 + a_1 b_0)}_{c_1} + \underbrace{(a_0 b_2 + a_1 b_1 + a_2 b_0)}_{c_2} + \dots + \underbrace{(a_0 b_n + \dots + a_n b_0)}_{c_n}$$

$$= a_0(\underbrace{b_0 + b_1 + b_2 + \dots + b_n}_{B_n}) + a_1(\underbrace{b_0 + b_1 + \dots + b_{n-1}}_{B_{n-1}}) + \dots + a_n(\underbrace{b_0}_{B_0})$$

$\beta_n = B_n - B$

$$= a_0(\underline{B} + \beta_n) + a_1(\underline{B} + \beta_{n-1}) + \dots + a_n(\underline{B} + \beta_0)$$

$$= \underbrace{(a_0 + a_1 + \dots + a_n)}_{A_n}\underline{B} + \underbrace{a_0\beta_n + a_1\beta_{n-1} + \dots + a_n\beta_0}_{\gamma_n}$$

$$= A_n \cdot B + \gamma_n$$

If we show that $\gamma_n \longrightarrow 0$, then it will follow

$$\sum_{n=0}^{+\infty} c_n = \lim_{n\to\infty} C_n = \lim_{n\to\infty} \left(A_n \cdot B + \gamma_n\right) = \left(\lim_{n\to\infty} A_n\right) B = A \cdot B$$

Indeed, let $\alpha = \sum_{n=0}^{+\infty} |a_n| < \infty$ (b/c $\sum a_n$ is assumed to converge absolutely)

Given $\varepsilon > 0$, since $\beta_n \longrightarrow 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N$

$\Rightarrow |\beta_n| < \varepsilon$.

$$|\gamma_n| = |\beta_0 a_n + \cdots + \beta_N a_{n-N} + \beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0|$$

triangle ineq

$$\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\underbrace{\beta_{N+1}} a_{n-N-1} + \cdots + \underbrace{\beta_n} a_0|$$

$|\beta_\bullet| < \varepsilon$

$$\le \left| \beta_0 a_n + \cdots + \beta_N a_{n-N} \right| + \varepsilon \cdot \alpha$$

Let $n \to +\infty$, Keeping N fixed; since $\underbrace{a_n \to 0 \text{ as } n \to \infty}_{(n^{th} \text{ term test})}$;

$$\limsup_{n \to \infty} |\gamma_n| \le \varepsilon \cdot \alpha$$

Since $\varepsilon > 0$ was arbitrary, $\gamma_n \longrightarrow 0$, as desired.

$\square$