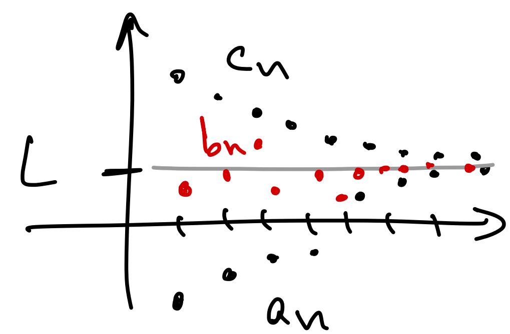
MAT320 Lecture 10 9/30/2020

Special sequences

Recall:

Squeeze Thm: If $a_n \le b_n \le c_n$ are sequences of real numbers, s.t. $a_n \to L$ and $c_n \to L$, then $b_n \to L$.



Pf: Since $a_n \to L$ and $c_n \to L$,

$\forall \varepsilon > 0 \quad \exists N, M \in \mathbb{N}$ s.t.

$$n \geq N \Rightarrow |a_n - L| < \varepsilon$$

$$\Leftrightarrow L - \varepsilon < a_n < L + \varepsilon \quad (1)$$

$$n \geq M \Rightarrow |c_n - L| < \varepsilon$$

$$\Leftrightarrow L - \varepsilon < c_n < L + \varepsilon \quad (2)$$

From (1), (2) and $a_n \le b_n \le c_n$, we have; for $n \geq \max\{M, N\}$, that;

$$L - \varepsilon \overset{(1)}{<} a_n \le b_n \le c_n \overset{(2)}{<} L + \varepsilon.$$

$\Rightarrow$ $|b_n - L| < \varepsilon$. This, by def., means that $b_n \to L$.

$\square$

<u>Thm:</u> (a) If $p > 0$, then $\frac{1}{n^p} \longrightarrow 0$

(b) If $p > 0$, then $\sqrt[n]{p} \longrightarrow 1$

(c) $\sqrt[n]{n} \longrightarrow 1$

(d) If $p > 0$, $\alpha \in \mathbb{R}$, $\frac{n^\alpha}{(1+p)^n} \longrightarrow 0$

All of these are as $n \to \infty$.

(e) If $|x| < 1$, then $x^n \longrightarrow 0$

Pf: (a) Given $\varepsilon > 0$, take $N \in \mathbb{N}$ s.t. $N > \left(\frac{1}{\varepsilon}\right)^{1/p}$.

Exists by the Archimedean prop.

Now if $n \geq N$, then

$$\left| \frac{1}{n^p} - 0 \right| = \frac{1}{n^p} < \frac{1}{N^p} = \varepsilon.$$

(b) If $p > 1$, set $b_n = \sqrt[n]{p} - 1$. Use the Squeeze Thm with $a_n \equiv 0$, b_n as above, and c_n as follows.

By the Binomial Thm, we have

$$\boxed{(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i}$$

$$\underbrace{1+\binom{n}{1}b_n}_{\substack{\shortparallel \\ 1+nb_n}} \le (1+b_n)^n = \left(1+\sqrt[n]{p}-1\right)^n = \left(\sqrt[n]{p}\right)^n = p.$$

$$\Longrightarrow \quad 1+nb_n \le p$$

So we get:

$$\begin{array}{ccccc} 0 & < & b_n & \le & \dfrac{p-1}{n} \\ \shortparallel & & \shortparallel & & \shortparallel \\ a_n & & b_n & & c_n \end{array}$$

Letting $c_n = \frac{p-1}{n}$, by the Squeeze Thm, since $a_n \to 0$, $c_n \to 0$ (by a), we have that $b_n \to 0$. The case $p=1$ is obvious, the case $p<1$ can be reduced to the above by using $\frac{1}{p}>1$. $\square$

c) Let $b_n := \sqrt[n]{n}-1 \ge 0$, and, by the Binomial thm:

$$n = \left(\sqrt[n]{n}\right)^n = (1+b_n)^n \ge \binom{n}{2} b_n^2 = \frac{n(n-1)}{2} b_n^2$$

$$\Rightarrow \quad 1 \geq \frac{n-1}{2} b_n^2 \Rightarrow \quad b_n \leq \sqrt{\frac{2}{n-1}} =: c_n$$

So letting $a_n = 0$, b_n as above and $c_n = \sqrt{\frac{2}{n-1}}$, by the Squeeze Thm, since $a_n \to 0$, $c_n \to 0$ (by a), we have that also $b_n \to 0$. $\square$

d) Let $k \in \mathbb{N}$ s.t. $k > \alpha$. For $n > 2k$, then:

$$(1+p)^n \overset{\text{Binomial Thm}}{>} \binom{n}{k} p^k = \frac{n!}{(n-k)!\,k!} p^k = \frac{\boxed{n}\boxed{(n-1)}\cdots\boxed{(n-k+1)}}{k!} p^k$$

$n - j > \frac{n}{2}$

$j = 0, \ldots, k-1$

$(k < n/2)$

$$> \left(\frac{n}{2}\right)^k \frac{1}{k!} p^k = \frac{n^k p^k}{2^k k!} = \frac{n^\alpha p^k}{2^k k!} \cdot n^{k-\alpha}$$

$\frac{n^\alpha}{(1+p)^n} \to 0$

From the above,

does not depend on n

$$\underbrace{0}_{a_n} < \underbrace{\frac{n^\alpha}{(1+p)^n}}_{b_n} < \underbrace{\frac{2^k k!}{p^k}\cdot n^{\alpha-k}}_{=c_n}$$

$k > \alpha \Rightarrow \alpha - k < 0$

$$\frac{1}{n^{k-\alpha}} \longrightarrow 0$$

By the Squeeze thm, with $a_n \le b_n \le c_n$ as above, since $a_n \to 0$, $c_n \to 0$, we have $b_n \to 0$, as desired.

e) Follows from previous item d), by setting $\alpha = 0$, $x = \frac{1}{1+p}$; then $x^n = \frac{1}{(1+p)^n} = \frac{n^0}{(1+p)^n} \longrightarrow 0$.

$\square$

Series: "Sum of <u>all</u> elements $\{a_n\}$ in a sequence".

$$\sum_{n=1}^{+\infty} a_n = a_1 + a_2 + \dots + a_n + \dots$$

Partial sums: $S_n = \sum_{k=1}^{n} a_k$ This a sequence $\{S_n\}$!

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + \dots \quad \dots + a_n$$

Def: $\sum a_n$ is identified with the sequence $\{S_n\}$.

$\sum a_n$ <u>converges</u> to L if and only if $S_n \longrightarrow L$.

(write $\sum a_n = L$). Similarly $\sum a_n$ <u>diverges</u> if $\{s_n\}$ diverges.

<u>Cauchy Convergence Theorem:</u> $\sum a_n$ converges if and only if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. if $m \geq n \geq N$, then

$$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

<u>Pf:</u> In $\mathbb{R}$, sequences are convergent if and only if they are Cauchy. Note that

$$|s_m - s_{n-1}| = \Big| \underbrace{(\cancel{a_1} + \cdots + \cancel{a_{n-1}} + a_n + \cdots + a_m)}_{s_m} - \underbrace{(\cancel{a_1} + \cdots + \cancel{a_{n-1}})}_{s_{n-1}} \Big|$$

$$= \left| \sum_{k=n}^{m} a_k \right| . \qquad \square$$

__Cor__: If $\sum a_n$ converges, then $|a_n| \longrightarrow 0$.

"nth term test"

__Pf__: Take $m = n$ in the above:

$$|S_n - S_{n-1}| = |a_n| .$$

__Note__: The converse is FALSE: $a_n = \frac{1}{n} \longrightarrow 0$

but $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ (harmonic series)

Comparison Test

Thm a) If $|a_n| \leq c_n$ for $n \geq N_0$, and $\sum c_n$ converges, then $\sum a_n$ converges

b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Pf: a) Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. if $m \geq n \geq N$

$$\left| \sum_{k=n}^{m} c_k \right| < \varepsilon \quad \text{(by Cauchy Conv. Thm)}.$$

Then:

$$\left| \sum_{k=n}^{m} a_k \right| \overset{\text{Triangle ineq.}}{\leq} \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \varepsilon$$

Again, by the Cauchy Conv. Thm, we have $\sum a_n$ converges.

b) is a consequence of a): If $\sum a_n$ converges, then apply a), and conclude that $\sum d_n$ must also converge (contradiction). $\square$

Geometric Series

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad x \in [0,1)$$

$\sum_{n=0}^{+\infty} x^n$ diverges otherwise, i.e., if $x \notin [0,1)$.

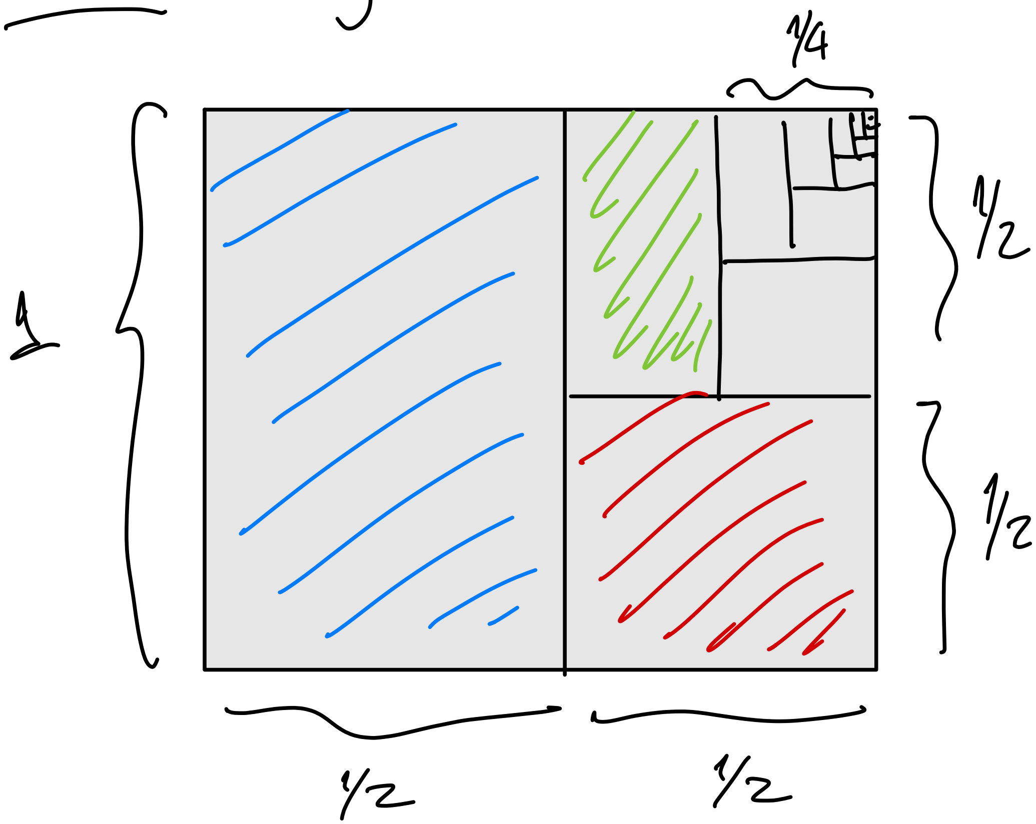
Pf: If $x \neq 1$, then $\underbrace{(1+x+x^2+\cdots+x^n)}_{S_n}(1-x) = 1+x+x^2+\cdots+x^n$
$-x-x^2-\cdots-x^{n+1}$
$= 1-x^{n+1}$

$$S_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \xrightarrow{n \nearrow +\infty} \frac{1}{1-x} \quad \text{if } x \in [0,1).$$

If $x=1$:

$$S_n = \sum_{k=0}^{n} 1 = n+1 \longrightarrow +\infty \quad \text{as} \quad n \to \infty.$$

<u>Bonus</u>: Why "Geometric" series?



$$\text{Area} = 1$$

$$= \underline{\frac{1}{2}} + \underline{\frac{1}{4}} + \underline{\frac{1}{8}} + \cdots$$

$$= \sum_{n=0}^{+\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \frac{1}{1-1/2} = 1$$

Thm: Suppose $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$. Then

$\sum_{n=1}^{+\infty} a_n$ converges if and only if $\sum_{k=0}^{+\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + \cdots$ converges.

Note:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots + a_8 + \cdots + a_{16} + \cdots$$

.2   .4   .8

$$\sum_{k=0}^{+\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots$$

Pf: Partial sums:

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$$t_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k} = \sum_{j=0}^{k} 2^j a_{2^j}$$

Note that b/c $a_n \geq 0$, the sequences $\{s_n\}$ and $\{t_k\}$ are monotonically increasing. Thus, they converge if and only if they are bounded. So, to prove thm, it suffices to show that:

<u>Claim</u>: $\{s_n\}$ is bounded $\iff$ $\{t_n\}$ is bounded.

For $n < 2^k$

$$
\begin{aligned}
s_n &= a_1 + a_2 + \cdots + a_n \\
&\leq a_1 + (a_2 + a_3) \cdots + a_n + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\
&\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k.
\end{aligned}
$$

Similarly, if $n > 2^k$ then

$$
\begin{aligned}
S_n &= a_1 + \cdots + a_{2^k} + \cdots + a_n \\
&\geq a_1 + \cdots + a_{2^k} \\
&= a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\
&\geq \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\
&= \frac{1}{2} t_k
\end{aligned}
$$

So it follows that $s_n \geq \frac{1}{2} t_k$. Altogether:

$$n < 2^k \Rightarrow s_n \leq t_k \,; \quad n > 2^k \Rightarrow 2s_n \geq t_k .$$

This proves the Claim that $\{s_n\}$ is bounded if and only if $\{t_k\}$ is bounded. $\square$

<u>Application</u>: "p-series"

<u>Thm</u>: $\sum \frac{1}{n^p}$ converges if $p > 1$

diverges if $p \leq 1$

<u>Pf</u>: If $p \leq 0$, then $\sum \frac{1}{n^p}$ diverges by the "n^{th} term test". If $p > 0$, then we may apply the previous thm, because

$$\frac{1}{n^p} \geq \frac{1}{(n+1)^p} \geq \frac{1}{(n+2)^p} \geq \cdots \geq 0.$$

Thus, by the Thm above,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff \sum_{k=0}^{+\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{+\infty} \left(2^{(1-p)}\right)^k$$

$$\| \leftarrow x = 2^{1-p}$$

$\sum_{k=0}^{\infty} x^k$ is a geometric series w/ $x = 2^{1-p}$

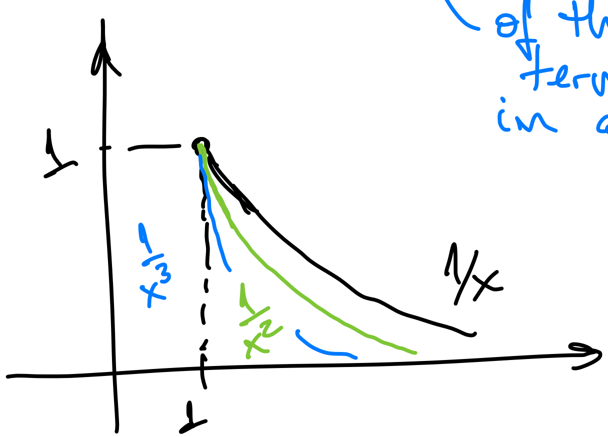
$$\iff X = 2^{1-p} < 1$$

$$\iff p > 1$$

Speed of decay and convergence

of the nth term a_n in a series

of that series.



$$f(x) = \frac{1}{x^p}$$

the larger p is, the faster $\frac{1}{n^p}$ decays to 0.

$p=1$ is the "boundary" between convergence & divergence.

$$\begin{cases} \sum \frac{1}{n^p} < \infty \iff p > 1 \\ \sum \frac{1}{n} = +\infty \quad \text{(harmonic series)} \end{cases}$$

Q: Can we "modify" the denom. of harmonic series to "make it" converge?

$$\sum_{n=1}^{\infty} \frac{1}{n \underbrace{\ldots}_{?}}$$

A: It will work if we insert any (positive) power of n, but can we add instead something that decays slower?

Yo!

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$ converges if $p>1$

diverges otherwise

$\log := \ln$ is the logarithm in base e.

$(\log e = 1)$

By Thm above w/ powers of 2:

$$a_n = \frac{1}{n(\log n)^p}, \quad a_{2^k} = \frac{1}{2^k(\log 2^k)^p} = \frac{1}{2^k(k\log 2)^p}$$

$$\sum_{n=2}^{+\infty} \frac{1}{n(\log n)^p} < \infty \iff \sum_{k=1}^{+\infty} 2^k a_{2^k} = \sum_{k=1}^{+\infty} \frac{1}{k^p(\log 2)^p} < +\infty$$

$$\frac{1}{(\log 2)^p} \sum_{k=1}^{+\infty} \frac{1}{k^p}$$

$\iff p > 1$ (by the p-Series)

Repeating the same reasoning:

$$\sum_{n=3}^{+\infty} \frac{1}{n(\log n)(\log \log n)^p} < +\infty \iff p > 1.$$

Upshot: There is no "natural boundary" on the rates of decay for a_n that corresponds to convergence of $\sum a_n$.