Solutions to Homework Set 7

1. Prove that the function $f(x) = \sum_{n=1}^{\infty} \frac{\cos(2020^n x^{2n})}{2^n}$ is continuous at every $x \in \mathbb{R}$.

   Hint: Use Video 6 of Lecture 23.

   Solution:

   The given function is $f(x) = \sum_{n=1}^{\infty} f_n(x)$, *where* $f_n(x) = \frac{\cos(2020^n x^{2n})}{2^n}$. *Clearly,* $|f_n(x)| \leq \frac{1}{2^n}$ *for all* $n \in \mathbb{N}$, *and* $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. *By Video 6 of Lecture 23, it follows that* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly. Therefore, since* $f(x)$ *is the uniform limit of continuous functions, it is continuous (see Video 2 of Lecture 24).*

2. Consider the sequence of functions $f_n \colon [-1,1] \to \mathbb{R}$, given by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

   (a) Find the pointwise limit of $f_n(x)$, i.e., compute $f_\infty(x) := \lim_{n\to\infty} f_n(x)$.

   (b) Find the pointwise limit of $f_n'(x)$, i.e., compute $g_\infty(x) := \lim_{n\to\infty} f_n'(x)$.

   (c) Prove that $f_n(x)$ converges uniformly to $f_\infty(x)$ on the interval $[-1,1]$.

   (d) Prove that $f_n'(x)$ does not converge uniformly to $g_\infty(x)$ on the interval $[-1,1]$.

   (e) Can you explain why $f_\infty'(x) = g_\infty(x)$ for all $x \neq 0$, but this fails for $x = 0$?

   Solution:

   (a) $f_\infty(x) := \lim_{n\to\infty} f_n(x) = \sqrt{x^2} = |x|$ *for all* $x \in [-1,1]$

   (b) $$g_\infty(x) := \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \begin{cases} 1 & \text{if } x \in (0,1], \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x \in [-1,0). \end{cases}$$

   (c) *Given* $\varepsilon > 0$, *let* $N \in \mathbb{N}$ *be such that* $N > \frac{1}{\varepsilon^2}$. *Then, for all* $x \in [-1,1]$, *and* $n \geq N$,

   $$|f_n(x) - f_\infty(x)| = \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} = \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \leq \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}}} \leq \frac{\frac{1}{n}}{\sqrt{\frac{1}{n}}} = \frac{1}{\sqrt{n}} < \varepsilon.$$

   Therefore, $f_n(x)$ *converges uniformly to* $f_\infty(x)$ *on the interval* $[-1,1]$.

   (d) *Since* $g_\infty(x)$, *which was computed in* (b), *is discontinuous at* $x = 0$, *it cannot be the uniform limit of the continuous functions* $f_n'(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}$, *cf. Video 2 of Lecture 24.*

(e) *The equality* $f'_\infty(x) = g_\infty(x)$ *holds for all* $x \neq 0$ *since (cf. Video 6 of Lecture 24), for any compact subset* $E \subset (0,1]$, *the sequence* $f'_n(x)$ *converges uniformly to* $g_\infty(x)$ *on* E*; similarly for compact subsets* $E \subset [-1,0)$*. However, this equality is not well-posed at* $x = 0$, *since* $f_\infty(x) = |x|$ *is not differentiable at* $x = 0$.

3. Suppose the functions $f_n \colon E \to \mathbb{R}$ are *uniformly* continuous, and converge uniformly to $f_\infty \colon E \to \mathbb{R}$. Prove that f_∞ is also *uniformly* continuous.

   Solution:

   Given $\varepsilon > 0$, *since* $f_n \colon E \to \mathbb{R}$ *converges uniformly to* $f_\infty \colon E \to \mathbb{R}$, *there exists* $N \in \mathbb{N}$ *such that if* $n \geq N$, *then for all* $p \in E$, *we have* $|f_n(p) - f_\infty(p)| < \frac{\varepsilon}{3}$. *Moreover, since* $f_N \colon E \to \mathbb{R}$ *is uniformly continuous, there exists* $\delta > 0$ *such that if* $x, y \in E$ *satisfy* $d(x,y) < \delta$, *then* $|f_N(x) - f_N(y)| < \frac{\varepsilon}{3}$. *Altogether, by the triangle inequality,*

   $$|f_\infty(x) - f_\infty(y)| \leq |f_\infty(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_\infty(y)| < \varepsilon$$

   for any $x, y \in E$ *with* $d(x,y) < \delta$. *Thus,* $f_\infty(x)$ *is uniformly continuous.*

4. Consider the function $f \colon (0,1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Does there exist a sequence of polynomials $p_n(x)$ that converges uniformly to $f \colon (0,1) \to \mathbb{R}$? Justify.

   Solution:

   No, there does not exist such a sequence $p_n(x)$ *of polynomials. If* $p_n(x)$ *are polynomials, then they define continuous functions* $p_n \colon [0,1] \to \mathbb{R}$. *Therefore, the uniform limit of* $p_n(x)$ *is also a continuous function* $\phi \colon [0,1] \to \mathbb{R}$, *by Video 2 of Lecture 24. In particular, if* $p_n(x)$ *converged uniformly to* $f \colon (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, *then* $f(x)$ *would admit a continuous (finite) extension to* $x = 0$, *which is a contradiction.*