Homework Set 7

DUE: DEC 2, 2020 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Prove that the function $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos\big(2020^n x^{2n}\big)}{2^n}$ is continuous at every $x \in \mathbb{R}$.

   Hint: Use Video 6 of Lecture 23.

2. Consider the sequence of functions $f_n \colon [-1,1] \to \mathbb{R}$, given by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

   (a) Find the pointwise limit of $f_n(x)$, i.e., compute $f_\infty(x) := \lim\limits_{n\to\infty} f_n(x)$.

   (b) Find the pointwise limit of $f_n'(x)$, i.e., compute $g_\infty(x) := \lim\limits_{n\to\infty} f_n'(x)$.

   (c) Prove that $f_n(x)$ converges uniformly to $f_\infty(x)$ on the interval $[-1,1]$.

   (d) Prove that $f_n'(x)$ does not converge uniformly to $g_\infty(x)$ on the interval $[-1,1]$.

   (e) Can you explain why $f_\infty'(x) = g_\infty(x)$ for all $x \neq 0$, but this fails for $x = 0$?

3. Suppose the functions $f_n \colon E \to \mathbb{R}$ are *uniformly* continuous, and converge uniformly to $f_\infty \colon E \to \mathbb{R}$. Prove that f_∞ is also *uniformly* continuous.

4. Consider the function $f \colon (0,1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Does there exist a sequence of polynomials $p_n(x)$ that converges uniformly to $f \colon (0,1) \to \mathbb{R}$? Justify.