Solutions to Homework Set 6

1. Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**.

   (a) There exists a monotonic function $f\colon [0,1] \to \mathbb{R}$ which is discontinuous at every point of the Cantor set P described in Lecture 6.

   (b) There exists a monotonic function $f\colon [0,1] \to \mathbb{R}$ which is continuous at every point of the Cantor set P described in Lecture 6.

   (c) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}\setminus\{0\}$, and $\lim\limits_{x\to 0} f'(x) = 2020$. Then $f(x)$ is also differentiable at $x = 0$ and $f'(0) = 2020$.

   (d) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous everywhere, differentiable on $\mathbb{R}\setminus\{0\}$, and $\lim\limits_{x\to 0} f'(x) = 2020$. Then $f(x)$ is also differentiable at $x = 0$ and $f'(0) = 2020$.

   (e) The function $\psi\colon [0,1] \to \mathbb{R}$ given by

   $$\psi(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ -1, & \text{if } x \notin \mathbb{Q}, \end{cases}$$

   is Riemann-Stieltjes integrable on $[0,1]$, i.e., $\psi \in \mathcal{R}(\alpha)$.

   (f) If a bounded function $f\colon [0,1] \to \mathbb{R}$ is such that f^2 is Riemann-Stieltjes integrable, then so is f; i.e., $f^2 \in \mathcal{R}(\alpha)$ implies $f \in \mathcal{R}(\alpha)$.

Solution:

(a) **FALSE:** *The set of discontinuities of a monotonic function is* countable *(Video 5 of Lecture 16), while the Cantor set P is* uncountable *(Video 4 of Lecture 6). Therefore, no such function exists.*

(b) **TRUE:** *Just take any monotonic function $f\colon [0,1] \to \mathbb{R}$ which is continuous at all points of $[0,1] \supset P$, such as $f(x) = x$.*

(c) **FALSE:** *For instance, consider the function $f\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} 2020x - 1, & \textit{if } x < 0 \\ 2020x + 1, & \textit{if } x > 0. \end{cases}$$

Clearly, $f(x)$ is differentiable on $\mathbb{R}\setminus\{0\}$, and $f'(x) = 2020$ for all $x \neq 0$, but $f(x)$ is not differentiable at $x = 0$.

(d) **TRUE:** *Under the stated assumptions, we may apply L'Hospital's rule (Video 2 of Lecture 18) and compute:*

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f'(x)}{1} = 2020.$$

Therefore $f(x)$ is differentiable at $x = 0$, and $f'(0) = 2020$.

(e) **FALSE:** *For any partition* $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ *of* $[0, 1]$*, we have*

$$m_i = \inf_{x \in [x_{i-1}, x_i]} \psi(x) = -1 \qquad \text{and} \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} \psi(x) = 1,$$

because every interval $[x_{i-1}, x_i]$ *contains both rational and irrational numbers. In particular, the lower and upper Riemann-Stieltjes sums for* ψ *are:*

$$L(P, \psi, \alpha) = \sum_{i=1}^{n} m_i \, \Delta\alpha_i = -\sum_{i=1}^{n} \Delta\alpha_i = -(\alpha(b) - \alpha(a))$$
$$U(P, \psi, \alpha) = \sum_{i=1}^{n} M_i \, \Delta\alpha_i = \sum_{i=1}^{n} \Delta\alpha_i = \alpha(b) - \alpha(a).$$

Therefore, for any partition P*, we have*

$$U(P, \psi, \alpha) - L(P, \psi, \alpha) = 2(\alpha(b) - \alpha(a)) > 0,$$

which makes $U(P, \psi, \alpha) - L(P, \psi, \alpha) < \varepsilon$ *impossible if we choose* $0 < \varepsilon < 2(\alpha(b) - \alpha(a))$*. Thus,* ψ *is not Riemann-Stieltjes integrable.*

(f) **FALSE:** *Take* $\psi \colon [0, 1] \to \mathbb{R}$ *to be the function from part (e). Clearly,* $\psi^2(x) = 1$ *for all* $x \in [0, 1]$ *and hence* $\psi^2 \in \mathcal{R}(\alpha)$*, while, by part (e), we have* $\psi \notin \mathcal{R}(\alpha)$*.*

2. Suppose $f \colon \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y| \, \phi(|x - y|),$$

where $\phi \colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\phi(0) = 0$. Prove that f must be a constant function.

Hint: Compute $f'(x)$ using the definition.

Solution:

Since $\phi(x)$ *is continuous and* $\phi(0) = 0$*, we have:*

$$|f'(x_0)| = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq \lim_{x \to x_0} \phi(|x - x_0|) = 0.$$

Therefore, $f(x)$ *is differentiable at all* $x \in \mathbb{R}$ *and* $f'(x) = 0$ *for all* $x \in \mathbb{R}$*. By the Mean Value Theorem, it follows that* $f(x) = f(y)$ *for all* $x, y \in \mathbb{R}$*, i.e.,* f *is a constant function.* □

3. Compute the Riemann-Stieltjes integral $\int_0^1 x^2 \, \mathrm{d}\alpha$, where $\alpha(x) = \begin{cases} 0, & \text{if } x \leq \frac{1}{2}, \\ 5, & \text{if } x > \frac{1}{2}. \end{cases}$

Solution:

By Video 6 of Lecture 20, letting $f(x) = x^2$*, we have:*

$$\int_0^1 x^2 \, \mathrm{d}\alpha = 5 f\left(\tfrac{1}{2}\right) = \frac{5}{4}.$$