Homework Set 5

DUE: NOV 4, 2020 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**.
   (Recall that a function $f\colon X \to Y$ is called *surjective*, or *onto*, if every point of Y belongs to its image, that is, if $f(X) = Y$.)
   (a) There exists a continuous function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$.
   (b) There exists a continuous surjective function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$.
   (c) There exists a continuous function $f\colon [0,1] \to \mathbb{R}$.
   (d) There exists a continuous surjective function $f\colon [0,1] \to \mathbb{R}$.
   (e) The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, is uniformly continuous.
   (f) The function $f\colon [0,1] \to \mathbb{R}$, $f(x) = x^2$, uniformly continuous.

2. Prove that a map $f\colon X \to Y$ between metric spaces is continuous if and only if the preimage $f^{-1}(C) \subset X$ of every closed subset $C \subset Y$ is closed in X.

3. Give a rigorous proof that the equation below has at least one real solution $x \in \mathbb{R}$.

$$x^{2020} + \frac{\pi}{1 + x^2 + \sin^2 x} = 10^{100}$$

   Please do not attempt to find the solution *explicitly*.