Solutions to Homework Set 4

1. Decide if each of the statements below is **true** or **false**. If it is true, give a complete **proof**; if it is false, give an explicit **counter-example**.

   (a) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

   (b) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n^2$ converges.

   (c) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

   (d) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for all $n \in \mathbb{N}$, then the power series $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for all $x \in [-1, 1]$.

Solution:

(a) **FALSE:** *Take* $a_n = \frac{1}{n^2}$*, so that* $\sqrt{a_n} = \frac{1}{n}$*, and recall that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

since these are p-series, but the first has $p > 1$ *while the second has* $p \leq 1$*.*

(b) **TRUE:** *Recall that*

$$(a_1 + \dots + a_n)^2 = a_1^2 + \dots + a_n^2 + 2\big(a_1 a_2 + \dots + a_{n-1} a_n\big),$$

therefore, since $a_j \geq 0$ *for all* $j \in \mathbb{N}$*, the partial sums satisfy*

$$0 \leq \sum_{k=1}^{n} a_k^2 \leq \left(\sum_{k=1}^{n} a_k \right)^2.$$

As $n \nearrow +\infty$*, the right-hand side converges to* S^2*, where* $S = \sum_{n=1}^{\infty} a_n < +\infty$*. Therefore the partial sums in the left-hand side are bounded. Since they form a monotonically increasing (and bounded) sequence, they converge; i.e.,* $\sum_{n=1}^{\infty} a_n^2$ *converges.* □

(c) **TRUE:** *Recall that for all* $A, B \in \mathbb{R}$*,*

$$0 \leq (A - B)^2 = A^2 - 2AB + B^2 \Longrightarrow AB \leq \frac{1}{2}(A^2 + B^2).$$

Using the above with $A = \sqrt{a_k}$ *and* $B = 1/k$*, we have the following inequalities:*

$$0 \le \sum_{k=1}^{n} \frac{\sqrt{a_k}}{k} \le \sum_{k=1}^{n} \frac{1}{2}\left(a_k + \frac{1}{k^2}\right) = \frac{1}{2}\sum_{k=1}^{n} a_k + \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k^2}.$$

Since $\sum_{n=1}^{\infty} a_n$ *converges, and so does the p-series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$*, it follows that the right-hand side in the above converges to a finite quantity. Thus, the partial sums* $\sum_{k=1}^{n} \frac{\sqrt{a_k}}{k}$ *form a monotonic increasing sequence which is bounded from above, and is hence convergent.* □

(d) **TRUE:** *Since* $a_j \ge 0$ *for all* $j \in \mathbb{N}$*, the partial sums satisfy, for any* $x \in [-1, 1]$*:*

$$\sum_{k=1}^{n} |a_k x^k| = \sum_{k=1}^{n} a_k |x|^k \le \sum_{k=1}^{n} a_k.$$

Since the right-hand side are partial sums of the convergent series $\sum_{n=1}^{\infty} a_n$*, it follows that the partial sums in the left-hand side also converge for any* $x \in [-1, 1]$*, i.e., the power series* $\sum_{n=1}^{\infty} a_n x^n$ *converges absolutely for all* $x \in [-1, 1]$*.* □

2. Use either the Root or Ratio test to find the radius of convergence of the following power series:

   (a) $\sum_{n=1}^{\infty} \frac{5^n}{n!} z^n$

   (b) $\sum_{n=1}^{\infty} \frac{7}{4^n} z^n$

   (c) $\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} z^n$

   Solution:

   (a) *Applying the Ratio test with* $a_n = \frac{5^n}{n!}$*, we have:*

$$\limsup_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to+\infty} \left|\frac{5^{(n+1)}}{(n+1)!}\frac{n!}{5^n}\right| = \lim_{n\to+\infty} \frac{5}{n+1} = 0.$$

Recall that the radius of convergence $0 \leq R \leq +\infty$ *of the power series* $\sum_{n=1}^{\infty} a_n z^n$ *satisfies*

$$\frac{1}{R} = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} \leq \limsup_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

and, since the right-hand side vanishes, it follows that the radius of convergence for the above power series is $R = +\infty$.

Note: This is the power series of the (analytic) function $f(z) = e^{5z}$.

(b) *Directly applying the Root test with* $a_n = \frac{7}{4^n}$, *we have that the radius of convergence* $0 \leq R \leq +\infty$ *of this power series satisfies:*

$$\frac{1}{R} = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} = \limsup_{n \to +\infty} \sqrt[n]{\left| \frac{7}{4^n} \right|} = \lim_{n \to +\infty} \frac{\sqrt[n]{7}}{4} = \frac{1}{4},$$

hence $R = 4$.

(c) *Directly applying the Root test with* $a_n = \frac{2^n}{\sqrt{n}}$, *we have that the radius of convergence* $0 \leq R \leq +\infty$ *of this power series satisfies:*

$$\frac{1}{R} = \limsup_{n \to +\infty} \sqrt[n]{|a_n|} = \limsup_{n \to +\infty} \sqrt[n]{\left| \frac{2^n}{\sqrt{n}} \right|} = \lim_{n \to +\infty} \frac{2}{\sqrt[2n]{n}} = 2,$$

hence $R = \frac{1}{2}$.