Solutions to Homework Set 2

1. Decide whether each of the following subsets of $\mathbb{R}$ is *countable* or *uncountable* and give a rigorous proof of your claim.

   (a) $\mathbb{R} \setminus \mathbb{Q} = \{x \in \mathbb{R} : x \text{ is irrational}\}$

   (b) $\mathbb{Q}(\sqrt{2}) = \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$

   Side note: $\mathbb{Q}(\sqrt{2})$ is a field! (But it does not have the least upper bound property.)

   Solution:

   (a) *The set $\mathbb{R}\setminus\mathbb{Q}$ of irrational numbers is* uncountable. *We proved in Lecture 3 (Video 6) that $\mathbb{Q}$ is countable, and in Lecture 6 (Video 2) that $\mathbb{R}$ is uncountable. If $\mathbb{R} \setminus \mathbb{Q}$ was countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would also be countable, because it would be a union of finitely many countable sets. Thus, $R \setminus \mathbb{Q}$ is uncountable.*

   (b) *The set $\mathbb{Q}(\sqrt{2})$ is* countable. *Indeed, there is a bijection $\phi\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}(\sqrt{2})$, given by $\phi(a, b) = a+b\sqrt{2}$, and we established in Lecture 3 (Video 5) that the (finite) cartesian product of countable sets, such as $\mathbb{Q} \times \mathbb{Q}$, is countable.*

2. Given three distinct natural numbers $a, b, c \in \mathbb{N}$, construct a bounded set $X \subset \mathbb{R}$ such that a, b, and c are the *only* limit points of X, and *none* of them belong to X.

   Solution:

   Let $X = \left\{a + \frac{1}{n} : n \in \mathbb{N}, n \geq 2\right\} \cup \left\{b + \frac{1}{m} : m \in \mathbb{N}, m \geq 2\right\} \cup \left\{c + \frac{1}{r} : r \in \mathbb{N}, r \geq 2\right\}$. Clearly, a is a limit point of X, since for any $\varepsilon > 0$, by the Archimedean property, there exists $n \in \mathbb{N}$, $n \geq 2$, such that $\frac{1}{n} < \varepsilon$, and hence $a + \frac{1}{n} \in (a, a + \varepsilon)$. Similarly, b and c are limit points of X. Aside from a, b, c, there is no other $x \in \mathbb{R}$ which is a limit point of X, since the distance from any $x \in \mathbb{R} \setminus (X \cup \{a, b, c\})$ to X is a (strictly) positive number. Finally, $X \cap \{a, b, c\} = \emptyset$ by construction, since $\{a, b, c\} \subset \mathbb{N}$ and $X \cap \mathbb{N} = \emptyset$.

3. Let (X, d) be any metric space. Prove that

   $$\overline{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X$$

   is also a distance function on X, i.e., prove that $(X, \overline{d})$ is also a metric space.

   Solution:

   We need to verify that $\overline{d}$ satisfies the following 3 properties:

   (i) $\overline{d}(x, y) > 0$ if $x \neq y$ and $\overline{d}(x, y) = 0$ if and only if $x = y$.

   Since $d(x, y) > 0$ if $x \neq y$, then also $\frac{d(x,y)}{1+d(x,y)} > 0$ if $x \neq y$. Moreover, $\frac{d(x,y)}{1+d(x,y)} = 0$ if and only if $d(x, y) = 0$, which holds if and only if $x = y$.

   (ii) $\overline{d}(x, y) = \overline{d}(y, x)$

This follows immediately from the assumption that $d(x,y) = d(y,x)$, namely:

$$\overline{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \overline{d}(y,x).$$

(iii) Triangle inequality: $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$

First, observe that if $a, b, c \geq 0$, then:

$$\begin{aligned}
a \leq b + c &\Longrightarrow a \leq b + c + 2bc + abc \\
&\Longrightarrow a + ab + ac + abc \leq (b + ab + bc + abc) + (c + ac + bc + abc) \\
&\Longrightarrow a(1+b)(1+c) \leq b(1+a)(1+c) + c(1+a)(1+b) \\
&\Longrightarrow \frac{a(1+b)(1+c)}{(1+a)(1+b)(1+c)} \leq \frac{b(1+a)(1+c)}{(1+a)(1+b)(1+c)} + \frac{c(1+a)(1+b)}{(1+a)(1+b)(1+c)} \\
&\Longrightarrow \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.
\end{aligned}$$

Applying the above with $a = d(x,z)$, $b = d(x,y)$, $c = d(y,z)$, it follows that the triangle inequality for d implies the triangle inequality for $\overline{d}$.

4. The *diameter* of a metric space (X, d) is defined to be:

$$\mathrm{diam}(X, d) := \sup \{d(x,y) : x, y \in X\}$$

Compute the following diameters, justifying your answer:

(a) $\mathrm{diam}(\mathbb{R}^n, d)$, where d is the usual (Euclidean) distance;

(b) $\mathrm{diam}(\mathbb{R}^n, \overline{d})$, where $\overline{d}$ is the distance defined in the previous exercise, with d still being the usual (Euclidean) distance.

Solution:

(a) $\mathrm{diam}(\mathbb{R}^n, d) = +\infty$, *i.e., the set $\{d(x,y) : x, y \in \mathbb{R}^n\}$ is not bounded from above. Indeed, if there existed an upper bound $M > 0$ such that $d(x,y) = ||x - y|| \leq M$ for all $x, y \in \mathbb{R}^n$, then taking $x = (0, \dots, 0)$ and $y = (M+1, 0, \dots, 0)$, we would have $||x - y|| = M + 1 > M$, a contradiction.*

(b) $\mathrm{diam}(\mathbb{R}^n, \overline{d}) = 1$. *First, note that $\overline{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} < 1$ for all $x, y \in \mathbb{R}^n$, so 1 is an upper bound for $\{\overline{d}(x,y) : x, y \in \mathbb{R}^n\}$. In order to show that 1 is the* least *upper bound for this set, suppose there exists $a < 1$ such that $\overline{d}(x,y) \leq a$ for all $x, y \in \mathbb{R}^n$. Then,*

$$\frac{d(x,y)}{1+d(x,y)} \leq a \quad\Longrightarrow\quad d(x,y) \leq a + a\,d(x,y) \quad\Longrightarrow\quad d(x,y) \leq \frac{a}{1-a},$$

so $M = \frac{a}{1-a}$ would be an upper bound for $\{d(x,y) : x, y \in \mathbb{R}^n\}$, contradicting what we proved above in (a). *Therefore, no such $a < 1$ exists, and $\sup\{\overline{d}(x,y) : x, y \in \mathbb{R}^n\} = 1$.*

5. Use the Heine–Borel Theorem to prove the following about compact sets in $\mathbb{R}^n$:

   (a) The union of finitely many compact sets in $\mathbb{R}^n$ is compact;

   (b) The intersection of any collection of compact sets in $\mathbb{R}^n$ is compact.

Solution:

Recall that, by the Heine–Borel Theorem, a subset of $\mathbb{R}^n$ *is* <u>*compact*</u> *if and only if it is* <u>*closed and bounded*</u>.

(a) *Let* $K_1, \ldots, K_\ell \subset \mathbb{R}^n$ *be compact sets, and* $K = \bigcup_{i=1}^{\ell} K_i$. *Since the union of finitely many closed sets is closed (Lecture 4, Video 6), it follows that* K *is closed. Moreover, as* K_i, $i = 1, \ldots, \ell$, *are bounded, there exist* $r_i > 0$, $i = 1, \ldots, \ell$, *such that* $K_i \subset B_{r_i}(0)$. *Setting* $r = \max_{1 \le i \le \ell} r_i$, *we have that* $K \subset B_r(0)$, *hence* K *is bounded. Thus,* K *is compact.*

(b) *Let* $K_\alpha \subset \mathbb{R}^n$, $\alpha \in A$, *be compact sets, and* $K = \bigcap_{\alpha \in A} K_\alpha$. *Since the intersection of arbitrarily many closed sets is closed (Lecture 4, Video 6), it follows that* K *is closed. Fix any* $\alpha \in A$. *Since* K_α *is bounded, there exists* $r > 0$ *such that* $K_\alpha \subset B_r(0)$. *Since* $K \subset K_\alpha$, *it follows that also* K *is bounded. Thus,* K *is compact.*