Homework Set 2

DUE: SEP 23, 2020 (VIA BLACKBOARD, BY 11.59PM)

To be handed in:

Please remember that all problems will be graded!

1. Decide whether each of the following subsets of $\mathbb{R}$ is *countable* or *uncountable* and give a rigorous proof of your claim.
   (a) $\mathbb{R} \setminus \mathbb{Q} = \{x \in \mathbb{R} : x \text{ is irrational}\}$
   (b) $\mathbb{Q}(\sqrt{2}) = \big\{a + b\sqrt{2} : a, b \in \mathbb{Q}\big\}$

   Side note: $\mathbb{Q}(\sqrt{2})$ is a field! (But it does not have the least upper bound property.)

2. Given three distinct natural numbers $a, b, c \in \mathbb{N}$, construct a bounded set $X \subset \mathbb{R}$ such that a, b, and c are the *only* limit points of X, and *none* of them belong to X.

3. Let (X, d) be any metric space. Prove that

$$\overline{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X$$

   is also a distance function on X, i.e., prove that $(X, \overline{d})$ is also a metric space.

4. The *diameter* of a metric space (X, d) is defined to be:

$$\operatorname{diam}(X, d) := \sup \{d(x, y) : x, y \in X\}$$

   Compute the following diameters, justifying your answer:
   (a) $\operatorname{diam}(\mathbb{R}^n, d)$, where d is the usual (Euclidean) distance;
   (b) $\operatorname{diam}(\mathbb{R}^n, \overline{d})$, where $\overline{d}$ is the distance defined in the previous exercise, with d still being the usual (Euclidean) distance.

5. Use the Heine–Borel Theorem to prove the following about compact sets in $\mathbb{R}^n$:
   (a) The union of finitely many compact sets in $\mathbb{R}^n$ is compact;
   (b) The intersection of any collection of compact sets in $\mathbb{R}^n$ is compact.