Solutions to Homework Set 1

1. Use the Archimedean property of $\mathbb{R}$ to rigorously prove that

$$\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0.$$

Remember that this entails proving 2 things:

- 0 is a *lower bound* for the set $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$;
- no real number larger than 0 is a lower bound for E, i.e., 0 is the *largest* possible lower bound.

Hint: I "argued" the above in Lecture 1 (Video 6), but, if you pay close attention, you will note that the Archimedean property must be used to make that rigorous.

Solution:

Claim 1: 0 is a lower bound for the set $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Proof of Claim 1: For all $n \in \mathbb{N}$, *we clearly have that* $\frac{1}{n} > 0$. □

Claim 2: No real number larger than 0 *is a lower bound for* E.

Proof of Claim 2: Suppose, by contradiction, that $x > 0$ *is a lower bound for* E. *By the Archimedean property, there exists* $n \in \mathbb{N}$ *such that* $nx > 1$. *Thus,* $x > \frac{1}{n} \in E$, *which contradicts the assertion that* x *is a lower bound for* E. □

From the above Claims 1 and 2, it follows that $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$. □

2. Let $A, B \subset \mathbb{R}$ be subsets bounded from below and from above, such that $A \subset B$. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Give examples to show that some (which?) inequalities above might be equalities even if A and B do not coincide.

Solution: First of all, all the quantities in the desired chain of inequalities exist because A *and* B *are bounded from below and from above. Since* $A \subset B$, every *lower bound for* B *is also a lower bound for* A. *Indeed, if* $\alpha \in \mathbb{R}$ *is such that* $\alpha \leq b$ *for all* $b \in B$, *then also clearly* $\alpha \leq a$ *for all* $a \in A$. *In particular, the* largest *lower bound,* $\inf B$, *for the set* B *is also a lower bound for* A. *Since* $\inf A$ *is the largest lower bound for* A, *it follows that* $\inf B \leq \inf A$. *Analogously, every upper bound for* B *is an upper bound for* A, *and so is the* least *such upper bound,* $\sup B$. *Since* $\sup A$ *is the smallest among the upper bounds for* A, *it follows that* $\sup A \leq \sup B$. *The middle inequality is obvious, since* $\inf A$ *is a lower bound for* A *while* $\sup A$ *is an upper bound for* A.

Finally, each one of the above inequalities may (individually) be an equality even if the sets do not coincide; e.g., consider the intervals $A = (0, 1)$ *and* $B = [0, 1]$. *Clearly,* $\inf A = \inf B = 0$ *and* $\sup A = \sup B = 1$, *but* $A \neq B$, *since* $0 \in B$ *but* $0 \notin A$. *The middle inequality can obviously be an equality without having* $A = B$, *e.g., take* $A = \{1/2\}$, $B = [0, 1]$.

3. A function $f\colon X \to \mathbb{R}$, $X \subset \mathbb{R}$, is called *bounded* if its image $\{f(x) : x \in X\}$ is a bounded set. In that case, we define $\sup f$ as its supremum, that is:

$$\sup f := \sup_{x \in X} f(x) = \sup\{f(x) : x \in X\}.$$

Prove each the following statements:

1. If $f, g\colon X \to \mathbb{R}$ are bounded functions, then so is their sum $(f+g)\colon X \to \mathbb{R}$;
2. $\sup(f+g) \leq \sup f + \sup g$;

Solution:

1. From the hypothesis that f and g are bounded, there exist $M, N \in \mathbb{R}$ such that $|f(x)| \leq M$ and $|g(x)| \leq N$, for all $x \in X$. In particular, $|(f+g)(x)| = |f(x)+g(x)| \leq |f(x)| + |g(x)| \leq M + N$ for all $x \in X$. Thus, $(f+g)\colon X \to \mathbb{R}$ is a bounded function.

2. By the above item, the image $A = \{f(x)+g(x) : x \in X\}$ of $(f+g)\colon X \to \mathbb{R}$ is bounded (and it is nonempty since $X \neq \emptyset$). Thus, $\sup(f+g) = \sup A$ exists. Define

$$B = \{f(x) + g(y) : x, y \in X\},$$

and note that $A \subset B$, so, by the previous exercise, $\sup A \leq \sup B$. It remains only to prove that $\sup B = \sup f + \sup g$. First, $\sup f + \sup g$ is an upper bound for B, since given $x, y \in X$, $f(x) + g(y) \leq \sup f + \sup g$ because $\sup f$ is an upper bound for all numbers of the form $f(x)$, $x \in X$, and $\sup g$ is an upper bound for all numbers $g(y)$, $y \in X$. Second, $\sup f + \sup g$ is the least *such upper bound. If not, then there would exist $\beta < \sup f + \sup g$ with $\beta \geq f(x) + g(y)$ for all $x, y \in X$. Let $r := \sup f + \sup g - \beta > 0$, and observe that $(\sup f) - \frac{r}{2} < \sup f$ is smaller than the smallest upper bound for the image of $f(x)$, so there exists $x_0 \in X$ such that $f(x_0) > (\sup f) - \frac{r}{2}$. Similarly, $(\sup g) - \frac{r}{2} < \sup g$ hence there exists $y_0 \in X$ such that $g(y_0) > (\sup g) - \frac{r}{2}$. Altogether,*

$$f(x_0) + g(y_0) > \sup f + \sup g - r = \beta,$$

which contradicts the above assertion that $\beta \geq f(x) + g(y)$ for all $x, y \in X$. This implies that $\sup f + \sup g$ is the least upper bound of B, so it is equal to $\sup B$, as desired. □

4. Give an example of functions f and g as in the previous exercise, such that only the *strict* inequality holds, i.e., $\sup(f+g) < \sup f + \sup g$.

Solution: Let $X = [-1, 1]$, $f(x) = x$, $g(x) = -x$. Clearly, $\sup f = \sup g = 1$ but $\sup(f+g) = 0$.