# Homework Set 1

DUE: SEP 9, 2020 (VIA BLACKBOARD, BY 11.59PM)

**To be handed in:**

*Please remember that all problems will be graded!*

1. Use the Archimedean property of $\mathbb{R}$ to rigorously prove that

$$\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0.$$

Remember that this entails proving 2 things:

- 0 is a *lower bound* for the set $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$;
- no real number larger than 0 is a lower bound for $E$, i.e., 0 is the *largest* possible lower bound.

Hint: I "argued" the above in Lecture 1 (Video 6), but, if you pay close attention, you will note that the Archimedean property must be used to make that rigorous.

2. Let $A, B \subset \mathbb{R}$ be subsets bounded from below and from above, such that $A \subset B$. Prove that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Give examples to show that some (which?) inequalities above might be equalities even if $A$ and $B$ do not coincide.

3. A function $f\colon X \to \mathbb{R}$, $X \subset \mathbb{R}$, is called *bounded* if its image $\{f(x) : x \in X\}$ is a bounded set. In that case, we define $\sup f$ as its supremum, that is:

$$\sup f := \sup_{x \in X} f(x) = \sup\{f(x) : x \in X\}.$$

Prove each the following statements:

   1. If $f, g\colon X \to \mathbb{R}$ are bounded functions, then so is their sum $(f+g)\colon X \to \mathbb{R}$;
   2. $\sup(f+g) \leq \sup f + \sup g$;

4. Give an example of functions $f$ and $g$ as in the previous exercise, such that only the *strict* inequality holds, i.e., $\sup(f+g) < \sup f + \sup g$.