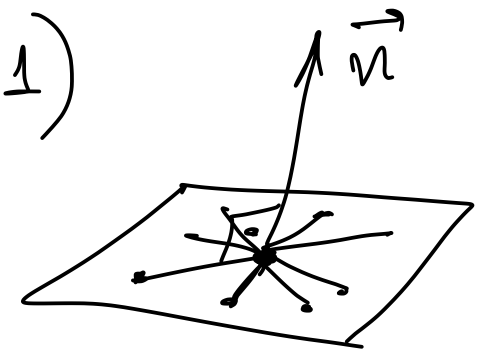
Q: Find the tangent plane to the graph of
$f(x,y) = x^3 + xy - y^2$ at $(1,1,1)$.

A: $F(x,y,z) = f(x,y) - z = x^3 + xy - y^2 - z$

$F(x,y,z) = 0 \iff z = f(x,y)$
(0-levelset of F) (graph (f))

$\nabla F(x,y,z) = (3x^2+y,\ x-2y,\ -1)$

$\nabla F(1,1,1) = (4,-1,-1) = \vec{n}$



Eqn of tangent plane:

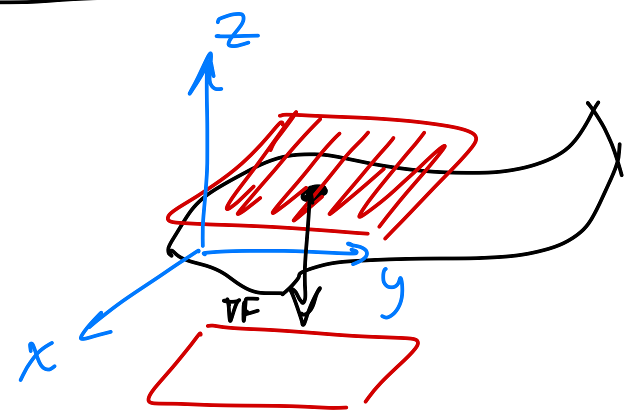
$\langle (x-1,\ y-1,\ z-1), (4,-1,-1) \rangle = 0$

$4(x-1) - (y-1) - (z-1) = 0$

$4x - 4 - y + 1 - z + 1 = 0$

$$\boxed{4x - y - z - 2 = 0}$$

Q: At what points is the tangent plane to the graph of $f(x,y) = x^3 + xy - y^2$ parallel to the xy-plane? (i.e., find the critical points of $f(x,y)$.)



A: $F(x,y,z) = f(x,y) - z$

$$\nabla F(x,y,z) = \underbrace{(3x^2+y,\ x-2y,\ -1)}_{\text{normal vector to tangent plane at } (x,y,f(x,y))}$$

$$\nabla F(x,y,z) = (0,0,-1) \iff \boxed{\nabla f(x,y) = 0} \iff \text{tangent plane parallel to } xy\text{-plane}$$

$$\begin{cases} 3x^2 + y = 0 \Rightarrow y = -3x^2 \\ x - 2y = 0 \Rightarrow x - 2(-3x^2) = 0 \end{cases}$$

$$x + 6x^2 = 0 \Rightarrow$$

$$x(1+6x) = 0 \Rightarrow x = 0 \text{ or } x = -\tfrac{1}{6}$$

- $x = 0$ and $y = 0$
- $x = -\frac{1}{6}$ and $y = -3\left(-\frac{1}{6}\right)^2 = -3 \cdot \frac{1}{36} = -\frac{3}{36} = -\frac{1}{12}$

There are two critical points:

$(0,0)$ and $\left(-\frac{1}{6}, -\frac{1}{12}\right)$.

The points in the graph are $(0,0,0)$, $\left(-\frac{1}{6}, -\frac{1}{12}, f\left(-\frac{1}{6}, -\frac{1}{12}\right)\right)$.