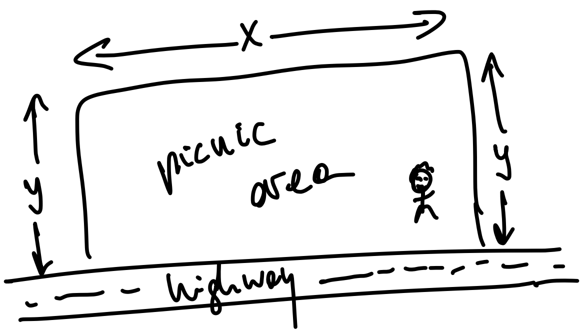
Lecture 28
(Review for Final Exam)

#1. The NYS Highway dept. is planning to build a picnic area along a highway, which will be rectangular with an area of 45,000 square yards, fenced off on the 3 sides not adjacent to the highway. What is the least amount of fence needed to complete the job?



Target function: $f(x,y) = x + 2y$

Constraint: $g(x,y) = xy = 45000$

Lagrange Multiplier: $\nabla f(x,y) = \lambda \nabla g(x,y)$

$$(1,2) = \lambda (y, x) \Rightarrow \begin{cases} 1 = \lambda y \\ 2 = \lambda x \end{cases} \Rightarrow \lambda = \frac{1}{y} = \frac{2}{x}$$

$$\Rightarrow \boxed{2y = x} \Leftrightarrow \boxed{y = \frac{x}{2}}$$

Plug back in the constraint: $x \cdot y = 45,000$

$$x \cdot \frac{x}{2} = 45,000$$

$$x^2 = 90,000 \Rightarrow \boxed{x = 300}$$

$$\Rightarrow \boxed{y = 150}$$

A: Minimum amount of fence needed is $f(300, 150) = 600$ yards.

#2 Consider the vector field $\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$\vec{F}(x,y,z) = (x+z, -y-z, x-y)$$

a) Is $\vec{F}$ conservative? If so, find a potential $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $\vec{F} = \nabla f$.

$\vec{F}: \underbrace{\mathbb{R}^3}_{\text{simply-connected}} \rightarrow \mathbb{R}^3$ is conservative if and only if $\text{curl}\, \vec{F} = 0$.

$$\text{curl}\, \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+z & -y-z & x-y \end{vmatrix} = (0,0,0).$$

Therefore $\vec{F}$ is <u>conservative</u>. Lets find a potential:

$$f(x,y,z) = \int x+z \, dx + g(y,z) = \frac{x^2}{2} + zx + g(y,z)$$

$$-y-z = \frac{\partial f}{\partial y}(x,y,z) = \frac{\partial g}{\partial y}(y,z) \Rightarrow g(y,z) = \int -y-z \, dy + h(z)$$

$$= -\frac{y^2}{2} - zy + h(z)$$

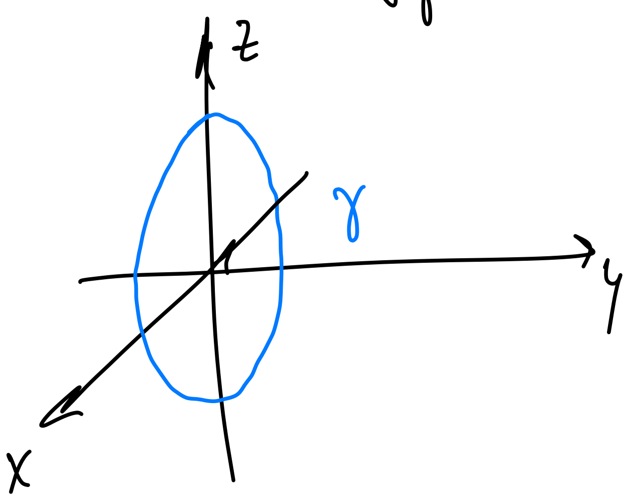
$$f(x,y,z) = \frac{x^2}{2} + zx - \frac{y^2}{2} - zy + h(z)$$

$$x-y = \frac{\partial f}{\partial z}(x,y,z) = x - y + h'(z) \Rightarrow h'(z) = 0$$

$$\Rightarrow h(z) = c.$$

$$\boxed{f(x,y,z) = \frac{x^2}{2} - \frac{y^2}{2} + z(x-y) + c}$$

b) Compute $\int_\gamma \vec{F}\, d\gamma$ where $\gamma(t) = (\cos t, 0, \sin t)$, $t \in [0, 2\pi]$



γ is a closed curve, so

$$\int_\gamma \vec{F}\, d\gamma = 0.$$

c) Compute $\int_\alpha \vec{F}\, d\alpha$ where $\alpha(t) = (t, t^2, t^3)$, $t \in [0,1]$.

$$\int_\alpha \vec{F}\, d\alpha = \int_\alpha \nabla f\, d\alpha = f(\alpha(1)) - f(\alpha(0)) = \underbrace{f(1,1,1)}_{=c} - \underbrace{f(0,0,0)}_{=c}$$

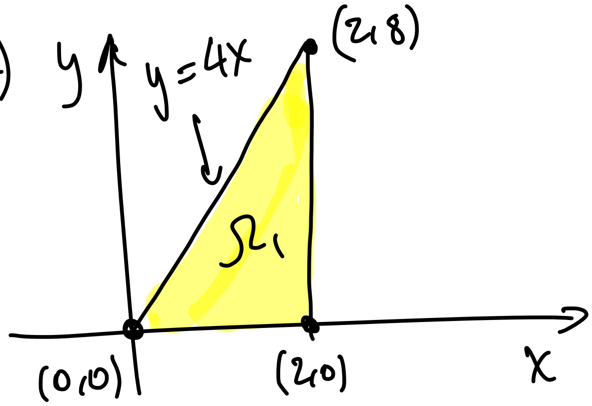
$$= 0.$$

#3 Compute $\iint_{\Omega_i} xy\, dA$ where

a) Ω_1 is the triangle with vertices $(0,0), (2,0), (2,8)$

b) Ω_2 is the disk of radius 2 centered at the origin.

a)



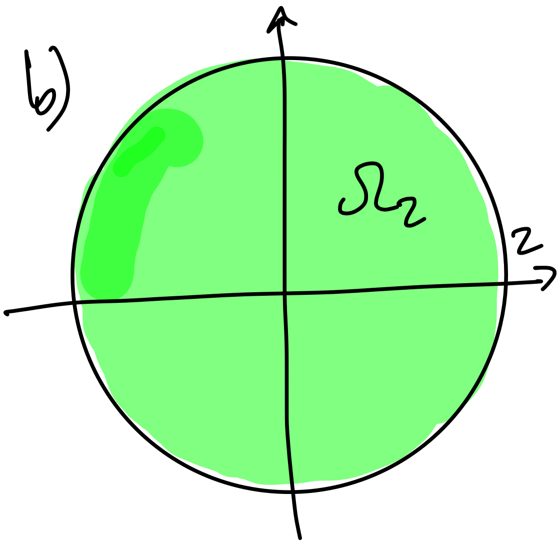
Ω_1: $0 \le x \le 2$
$0 \le y \le 4x$

$$\iint_{\Omega_1} xy\, dA = \int_0^2 \int_0^{4x} xy\, dy\, dx =$$

$$= \int_0^2 \frac{xy^2}{2}\Big|_0^{4x} dx = \int_0^2 \frac{16x^3}{2}\, dx = 8\left(\frac{x^4}{4}\right)\Big|_0^2 = 2.2^4$$

$$= \boxed{32.}$$

b)



Ω_2: $0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$\iint_{\Omega_2} xy\, dA = \int_0^{2\pi} \int_0^2 r^2 \cos\theta \sin\theta\, r\, dr\, d\theta$$

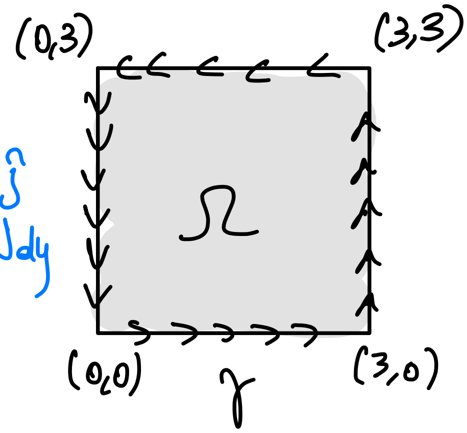
$$= \Bigg(\underbrace{\int_0^{2\pi} \frac{\sin 2\theta}{2}\, d\theta}_{=0.}\Bigg)\Bigg(\underbrace{\int_0^2 r^3\, dr}_{= \frac{r^4}{4}\big|_0^2 = 4.}\Bigg) = \boxed{0.}$$

#4. Use Green's Theorem to compute the following:

a) $\int_\gamma \underbrace{\sqrt{1+x^3}}_{M}\, dx + \underbrace{2xy}_{N}\, dy$

$\vec{F} = (M,N)$
$= M\hat{\imath} + N\hat{\jmath}$
$= M\,dx + N\,dy$

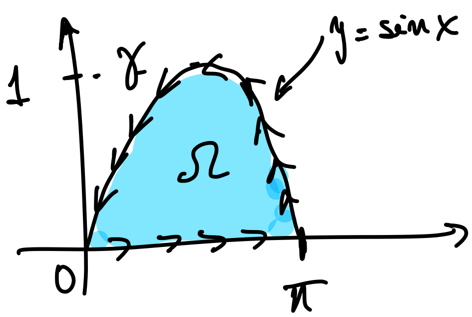
γ: boundary of square with vertices $(0,0), (3,0), (0,3), (3,3)$. oriented counterclockwise



$$\int_\gamma M\,dx + N\,dy = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^3\int_0^3 2y - 0 \quad dx\,dy$$

$$= 2.3.\int_0^3 y\,dy = 6.\frac{y^2}{2}\Big|_0^3 = 3.3^2 = \boxed{27}$$

b) $\int_\gamma \big(\underbrace{xy + e^{x^2}}_{M}\big)\,dx + \big(\underbrace{x^2 - \ln(1+y)}_{N}\big)\,dy = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA =$



Ω: $0 \le x \le \pi$
$0 \le y \le \sin x$

$$= \int_0^\pi \int_0^{\sin x} (2x - x)\,dy\,dx$$

$$= \int_0^\pi xy\Big|_0^{\sin x} dx = \int_0^\pi x\sin x\,dx$$

$$\overset{\text{parts}}{=} (-x\cos x + \sin x)\Big|_0^\pi = \boxed{\pi}$$

#5 Consider the function $f(x,y) = 4x^3 - 3y^2 + 12xy - 18y + 400$.

a) Find the tangent plane to the graph of $f(x,y)$ at the point $(0,0,f(0,0))$

b) Find all critical points of $f(x,y)$.

c) Classify all critical points into local min/max, saddle.

$$\nabla f(x,y) = \left(12x^2 + 12y, -6y + 12x - 18\right).$$

a) $f(0,0) = 400$. Graph of f is the zero levelset of

$$F(x,y,z) = f(x,y) - z$$

$$\nabla F(x,y,z) = \left(12x^2 + 12y, -6y + 12x - 18, -1\right)$$

$$\nabla F(0,0,400) = (0, -18, -1)$$

Tangent plane to graph of f at $(0,0,400)$ is

$$\langle (x-0, y-0, z-400), (0,-18,-1) \rangle = 0.$$

$$+18y + (z-400) = 0 \Rightarrow \boxed{18y + z = 400}$$

b) $\nabla f(x,y) = \left(12x^2 + 12y, -6y + 12x - 18\right) = (0,0)$

$$\begin{cases} x^2 + y = 0 \\ -y + 2x - 3 = 0 \Rightarrow \boxed{y = 2x - 3} \end{cases} \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow \begin{cases} x = 1 \\ x = -3 \end{cases}$$

If $x=1$, $y = 2(1)-3 = -1$

If $x=-3$, $y = 2(-3)-3 = -9$.

$\boxed{\text{Critical points: } (1,-1),\ (-3,-9).}$

c) $\nabla f(x,y) = \left(12x^2+12y,\ -6y+12x-18\right)$.

$$\operatorname{Hess} f(x,y) = \begin{bmatrix} 24x & 12 \\ 12 & -6 \end{bmatrix}$$

$$\operatorname{Hess} f(1,-1) = \begin{pmatrix} 24 & 12 \\ 12 & -6 \end{pmatrix}$$

$\det \operatorname{Hess} f(1,-1) = -6 \cdot 24 - 144 < 0$

$\Longrightarrow \boxed{(1,-1) \text{ is a saddle}}$

$$\operatorname{Hess} f(-3,-9) = \begin{pmatrix} -72 & 12 \\ 12 & -6 \end{pmatrix}$$

$\det \operatorname{Hess} f(-3,-9) = 432 - 144 > 0$

$\frac{\partial^2 f}{\partial x^2}(-3,-9) = -72 < 0$

$\Longrightarrow \boxed{(-3,-9) \text{ is a local max.}}$