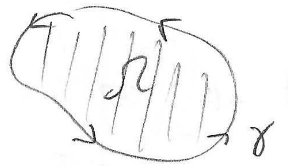
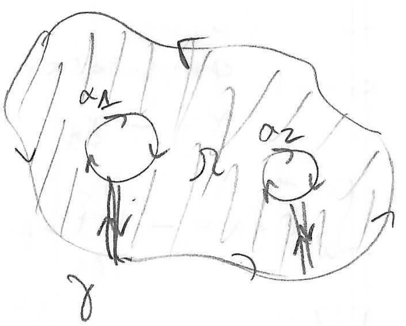
Green's Thm: Ω simply-connected, $\vec{F} = (P,Q) = P\,dx + Q\,dy$.



$$\int_\gamma \vec{F}\, d\gamma = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Q: what about non-simply-connected regions?



$$\int_\gamma \vec{F}\, d\gamma + \int_{\alpha_1} \vec{F}\, d\alpha_1 + \int_{\alpha_2} \vec{F}\, d\alpha_2 = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

outside curve: γ, oriented counter-clockwise

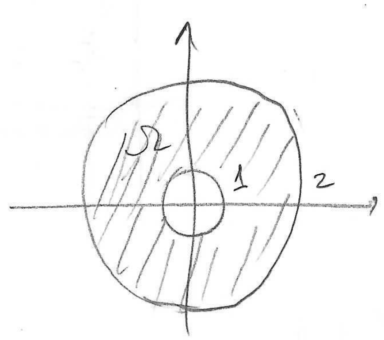
inside curves: α_i, oriented clockwise ⚠

In general:

$$\int_\gamma \vec{F}\, d\gamma + \sum_{i=1}^{k} \int_{\alpha_i} \vec{F}\, d\alpha_i = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

"Green's Thm with holes"

example.



$$\vec{F}(x,y) = (x+y, 1)$$

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_\Omega -1\, dA = -\text{Area}(\Omega) = -(4\pi - \pi) = \boxed{-3\pi}$$

$\gamma_R(t) = (R\cos t, R\sin t)$ circle of radius R.

$$\int_{\gamma_R} \vec{F}\, d\gamma_R = \int_0^{2\pi} \langle (R(\cos t + \sin t), 1), (-R\sin t, R\cos t) \rangle \, dt$$

$$= \int_0^{2\pi} -R^2 \sin t \cos t - R^2 \sin^2 t + R\cos t \; dt$$

$$= -R^2 \underbrace{\int_0^{2\pi} \frac{\sin 2t}{2}\, dt}_{=0} - R^2 \int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt + R\underbrace{\int_0^{2\pi} \cos t\, dt}_{=0}$$

$$= -R^2 \left. \frac{t}{2} \right|_0^{2\pi} = -\pi R^2.$$

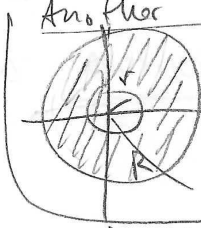
$$\int_{\gamma_2} \vec{F}\, d\gamma_2 = -4\pi \qquad \boxed{\int_{\gamma_1} \vec{F}\, d\gamma_1 = -\pi} \longleftarrow$$

Reverse orientation: $\alpha = -\gamma_1$

$$\int_\alpha \vec{F}\, d\alpha = -\int_{\gamma_1} \vec{F}\, d\gamma_1 = \pi$$

$$\underbrace{\int_{\gamma_2} \vec{F}\, d\gamma_2}_{-4\pi} + \underbrace{\int_\alpha \vec{F}\, d\alpha}_{\pi} = \underbrace{\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA}_{-3\pi} \quad \checkmark$$

Another nice example



$F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$

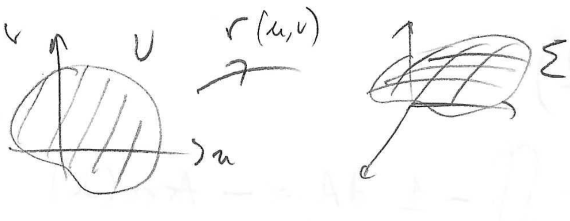
$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \; \dots$

$$\int_{S^1_r} \vec{F}\, d\gamma = 2\pi \quad \forall r$$

A quick tour of surface integrals:

$r : U \subset \mathbb{R}^2 \longrightarrow \Sigma \subset \mathbb{R}^3$

$r(u,v) = (x(u,v), y(u,v), z(u,v))$

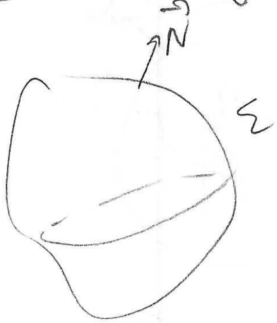


$$\iint_\Sigma f\, dS = \iint_U f(r(u,v)) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA.$$

$$\longleftarrow \left\| \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \right\|$$

Flux integral: $\vec{F}$ = vector field,

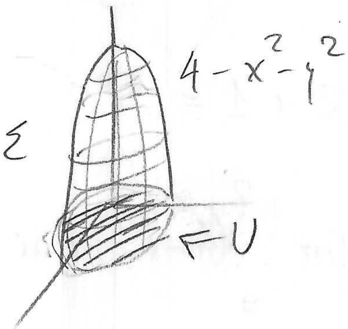
$\vec{N}$ = (unit) normal vector to Σ.



$$\iint_\Sigma \langle \vec{F}, \vec{N} \rangle \, dS = \begin{pmatrix} \text{flux of } \vec{F} \\ \text{across } \Sigma. \end{pmatrix}$$

compute as:
$$\frac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}$$

Ex: $F(x,y,z) = (2x, 2y, z)$



$r(u,v) = (u, v, 4-u^2-v^2)$

U = (disk of radius 2)

Unit normal vector: $\vec{N} = \dfrac{\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}}{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}$

$$\iint_\Sigma \langle \vec{F}, \vec{N} \rangle \, dS = \iint_U \langle \vec{F}, \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \rangle \, dA,$$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = (+2u, +2v, 1)$$

$$= \iint_U \langle (2u, 2v, 4-u^2-v^2), (2u, 2v, 1) \rangle \, dA$$

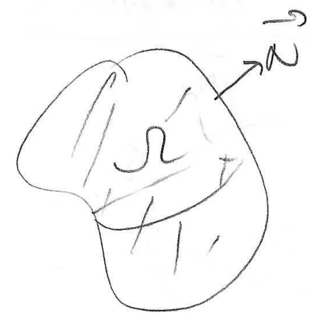
$$= \iint_U 4u^2 + 4v^2 + 4 - u^2 - v^2 \, dA$$

$$= \iint_U 3(u^2+v^2) + 4 \, dA = \int_0^{2\pi} \int_0^2 (3r^2+4) r \, dr \, d\theta$$

$$= 2\pi \int_0^2 3r^3 + 4r \, dr = 2\pi \left(\frac{3r^4}{4} + 2r^2 \right) \Big|_0^2$$

$$= 2\pi (12 + 8) = 40\pi.$$

Gauss thm / Divergence thm:



$$\underbrace{\iint_\Sigma \langle \vec{F}, \vec{N} \rangle \, dS}_{\text{flux of } \vec{F} \text{ accross } \Sigma} = \underbrace{\iiint_\Omega \operatorname{div} \vec{F} \, dV}_{\text{triple integral of } \operatorname{div} \vec{F}}$$

in the previous example:

$$\vec{F}(x,y,z) = (2x, 2y, z) \qquad \operatorname{div} \vec{F} = 2 + 2 + 1 = 5.$$

$$\iiint_\Omega 5 \, dV = 5 \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r \, dz \, dr \, d\theta = 10\pi \int_0^2 (4 - r^2) r \, dr$$

$$= 10\pi \int_0^2 4r - r^3 \, dr = 10\pi \left(2r^2 - \frac{r^4}{4} \right) \Big|_0^2 = 10\pi (8 - 4)$$

$$= 40\pi$$

I slightly lied:
what is missing? The bottom!

$$\iint_U \langle \vec{F}, \vec{N} \rangle \, dS = \int_0^{2\pi} \int_0^2 \langle (2r\cos\theta, 2r\sin\theta, 0), (0,0,-1) \rangle \, r \, dr \, d\theta$$

$\vec{N} = (0,0,-1)$

$$= \int_0^{2\pi} \int_0^2 0 \cdot r \, dr \, d\theta = 0$$

so flux accross the bottom is zero anyways...

Stokes:

$$\int_\gamma \vec{F} \, d\gamma = \iint_\Sigma \langle \nabla \times \vec{F}, \vec{N} \rangle \, dS$$

flux integral of curl of $\vec{F}$