Recap from last time.

<u>Thm.</u> TFAE:

1. $\vec{F}$ is conservative (i.e., $\vec{F} = \nabla f$ for some $f$)
2. $\int_\gamma \vec{F}\, d\gamma$ depends only on <u>endpoints</u> of $\gamma$
3. $\int_\gamma \vec{F}\, d\gamma = 0$ if $\gamma$ is closed.

<u>Thm</u>: If $\Omega \subset \mathbb{R}^n$ $(n=2,3)$ is simply-connected, then

$\vec{F} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is conservative $\iff$ $\nabla \times \vec{F} = 0$ $(n=3)$

$$\boxed{\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad (n=2)}$$

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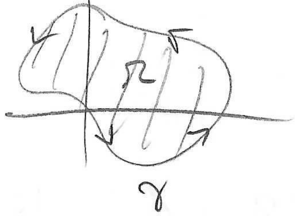
<u>Today</u>: Green's Thm

<u>Thm.</u> Suppose $\Omega$ is simply-connected and denote by $\gamma$ its boundary (with counter-clockwise orientation). Then, if $\vec{F} = (P, Q)$ is a vector field defined on $\Omega$, we have

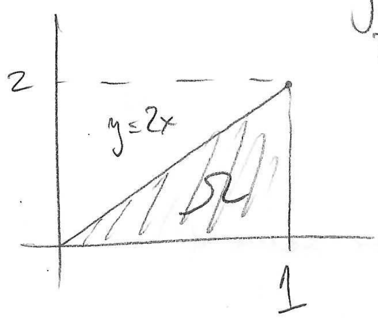



$$\int_\gamma \vec{F}\, d\gamma = \iint_\Omega \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

line integral of vector field

double integral (of function).

Ex:




$$\int_\gamma xy\,dx + x^2y^3\,dy = \iint_\Omega 2xy^3 - x\,dA$$

$$= \int_0^1 \int_0^{2x} 2xy^3 - x\,dy\,dx$$

$$= \int_0^1 8x^5 - 2x^2\,dx = \frac{4}{3}x^6 - \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3}$$

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}$$

$\gamma_1(t) = (t,0)$

$\gamma_2(t) = (1, 2t)$

$\gamma_3(t) = (1-t, 2-2t)$

$$\int_{\gamma_1} \vec{F} = \int_0^1 \langle (0,0), \dots \rangle = 0$$

$$\int_{\gamma_2} \vec{F} = \int_0^1 \langle (2t, 8t^3), (0,2) \rangle\,dt$$

$$= \int_0^1 16t^3\,dt = 4t^4 \Big|_0^1 = 4.$$

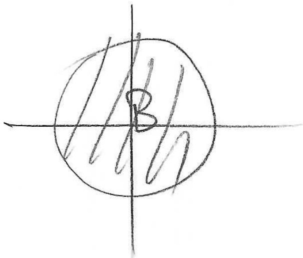
$$\int_{\gamma_3} \vec{F} = \int_0^1 \langle ((1-t)(2-2t), (1-t)^2(2-2t)^3), (-1,-2) \rangle$$

$$= \int_0^1 -2(1-t)^2 - 2(1-t)^5 \cdot 8\,dt$$

$$= \int_0^1 -2(1-t)^2 - 16(1-t)^5\,dt \qquad 1-t = u, \quad -dt = du$$

$$= \int_1^0 +2u^2 + 16u^5\,du = \frac{2u^3}{3} + \frac{16u^6}{6} \Big|_1^0 = -\frac{2}{3} - \frac{8}{3} = -\frac{10}{3}$$

$$\Rightarrow \int_\gamma = 4 - \frac{10}{3} = \frac{12-10}{3} = \frac{2}{3}$$




$$\vec{F} = \left(xy + 1,\ y^2 + e^y\right)$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - x$$

$$\iint_B \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \int_0^{2\pi}\int_0^1 - r\cos\theta \; r\,dr\,d\theta = 0$$

$$\int_{S^1} P\,dx + Q\,dy = \int_0^{2\pi} - \sin t + \cos t\, e^{\sin t}\, dt = 0.$$

<u>But</u>: $\vec{F}$ is <u>not</u> conservative!