Lecture 23

<u>Recall</u>: Line integral of $\vec{F}$ along path $\gamma(t)$, $t\in[a,b]$

$$\int_\gamma \vec{F}\, d\gamma = \int_a^b \langle \vec{F}(\gamma(t)), \gamma'(t)\rangle\, dt$$

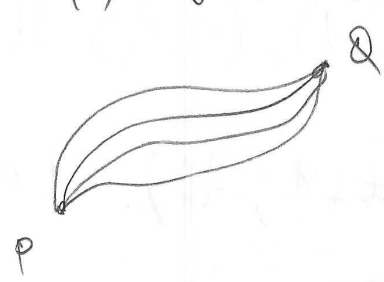
<u>Fundamental thm of Calculus for Line Integrals</u>: If $\vec{F} = \nabla f$ is a conservative vector field, then

$$\int_\gamma \vec{F}\, d\gamma = f(\gamma(b)) - f(\gamma(a)) \qquad (*)$$

<u>Pf</u>:
$$\int_\gamma \vec{F}\, d\gamma = \int_a^b \langle F(\gamma(t)), \gamma'(t)\rangle\, dt$$
$$= \int_a^b \langle \nabla f(\gamma(t)), \gamma'(t)\rangle\, dt$$
$$\underset{\text{(Chain Rule)}}{=} \int_a^b \frac{d}{dt} f(\gamma(t))\, dt = f(\gamma(b)) - f(\gamma(a)).$$
$\square$

Sometimes, if $\gamma(b) = Q$, $\gamma(a) = P$, we write

$$\int_P^Q \vec{F}\, d\gamma$$



$$\boxed{\text{If } \vec{F} \text{ is conservative, this Integral DOES NOT depend on the path } \gamma\text{, just on its endpoints!}}$$

(b/c the RHS of $(*)$ is like that!

Example. $\vec{F}(x,y) = (xy^2, x^2y)$ conservative? [HW prob]

$$\frac{\partial}{\partial y}(xy^2) = \frac{\partial}{\partial x}(x^2y) = 2xy \checkmark$$

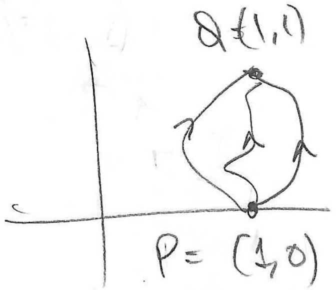
$\Omega = \mathbb{R}^2$ is simply-connected $\checkmark$ $\Rightarrow$ Yes!

$$f(x,y) = \int xy^2\,dx + g(y) = \frac{x^2y^2}{2} + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{\cancel{2}x^2y}{\cancel{2}} + g'(y) = x^2y \Rightarrow g'(y) = 0 \Rightarrow g(y) = C.$$

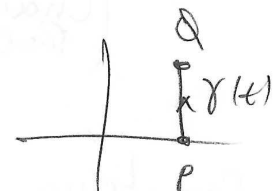
$$\boxed{f(x,y) = \frac{x^2y^2}{2} + C}$$

(Say we choose $C=0$)



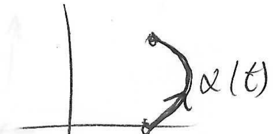
$$\int_\gamma \vec{F}\,d\gamma = f(1,1) - f(1,0) = \frac{1}{2} - 0 = \frac{1}{2}.$$

E.g. $\gamma(t) = (1,t)$, $t \in [0,1]$.



$$\int_\gamma \vec{F}\,d\gamma = \int_0^1 \langle (t^2, t), (0,1) \rangle\,dt = \int_0^1 t\,dt = \frac{1}{2}.$$

Eg. $\alpha(t) = (-t^2+t+1, t)$ $t \in [0,1]$



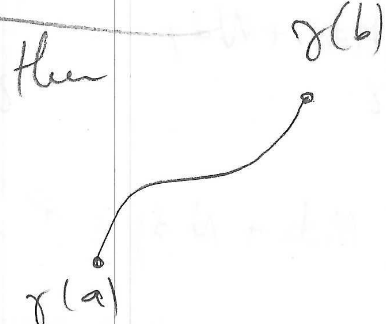
$$\int_\alpha \vec{F}\,d\alpha = \int_0^1 \langle ((-t^2+t+1)t^2, (-t^2+t+1)^2 t), (-2t+1, 1) \rangle\,dt$$

$$= \int_0^1 \langle (-t^4+t^3+t^2, t^5-2t^4-t^3+2t^2+t), (-2t+1, 1) \rangle\,dt =$$

Fundamental Thm of Line Integrals

If $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla f$, then

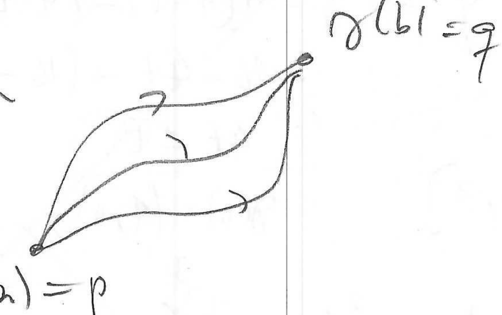
$$\int_\gamma \vec{F}\, d\gamma = f(\gamma(b)) - f(\gamma(a))$$



__Pf.:__ Compute + Chain Rule

$$\int_\gamma \vec{F} d\gamma = \int_a^b \langle \nabla f(\gamma(t)), \gamma'(t) \rangle\, dt = \int_a^b \frac{d}{dt}(f \circ \gamma)(t) = f(\gamma(b)) - f(\gamma(a)).$$

__Cor:__ Independence of path



$$\int_p^q \vec{F}$$

__TFAE:__

1) $\vec{F} = \nabla f$

2) $\int_\gamma F\, d\gamma$ indep. of path γ (only endpoints)

3) $\oint_\gamma F\, d\gamma = 0 \quad \forall \gamma$ __closed__