Recall: $\vec{F} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ vector field

$\gamma : [a,b] \to \Omega$ smooth curve

$$\int_\gamma \vec{F}\, d\gamma = \int_a^b \langle \vec{F}(\gamma(t)), \gamma'(t) \rangle\, dt. = \text{Work done by the force } \vec{F} \text{ when moving along trajectory } \gamma$$

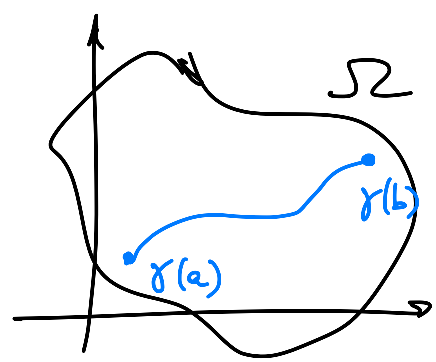
Fundamental Theorem of Calculus (Baby version):

$$\int_a^b f(t)\, dt = F(b) - F(a) \quad \text{where } F'(t) = f(t)$$

Fundamental Theorem of Calculus (Line integrals):

Let $\vec{F} = \nabla \phi$ be a conservative vector field. Then given any smooth curve $\gamma : [a,b] \to \Omega$, we have:

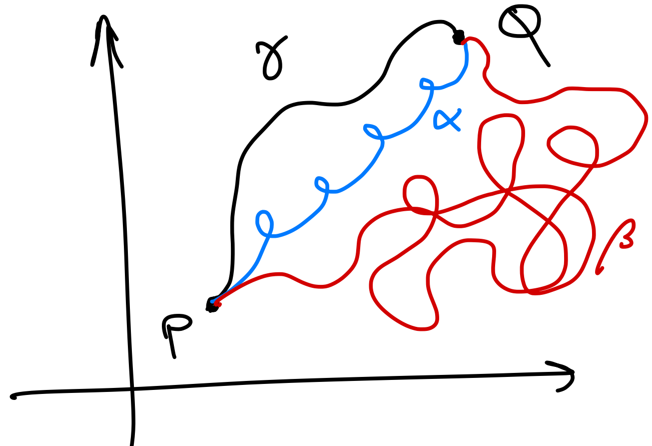
$$(*) \quad \int_\gamma \vec{F}\, d\gamma = \phi(\gamma(b)) - \phi(\gamma(a))$$



Very important to observe:

RHS of $(*)$ does not depend on the path γ, but only on its endpoints $\gamma(a), \gamma(b)$

Cor: If $\vec{F}$ is conservative, then $\int_\gamma \vec{F}\,d\gamma$ depends only on the endpoints of γ and not on γ itself.



$$\underbrace{\int_\gamma \vec{F}\,d\gamma}_{} = \int_\alpha \vec{F}\,d\alpha = \int_\beta \vec{F}\,d\beta$$

All equal!

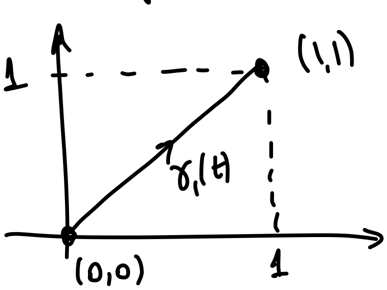
Often the above quantity is denoted $\int_P^Q \vec{F}$.

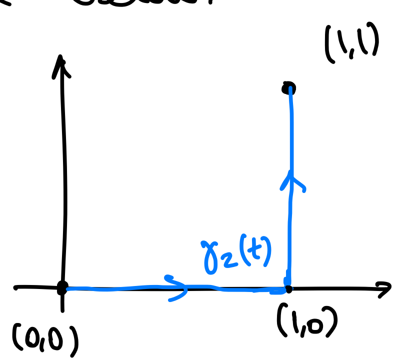
<u>Pf of the F.T.C. (Line integrals) (*)</u>: If $\vec{F} = \nabla\phi$, then

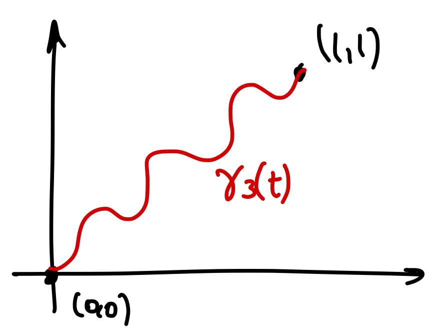
$$\int_\gamma \vec{F}\,d\gamma = \int_\gamma \nabla\phi\,d\gamma = \int_a^b \underbrace{\langle \nabla\phi(\gamma(t)), \gamma'(t)\rangle}_{= \frac{d}{dt}\phi(\gamma(t))}\,dt$$

$$= \int_a^b \frac{d}{dt}\phi(\gamma(t))\,dt \overset{\text{FTC (Baby version)}}{=} \phi(\gamma(b)) - \phi(\gamma(a)). \qquad \square$$

<u>Ex:</u> Find the line integral of $\vec{F}(x,y) = (xy^2, x^2y)$ along the paths γ_i shown below:







To see that $\vec{F}(x,y) = (\underset{\overset{\parallel}{M}}{xy^2}, \underset{\overset{\parallel}{N}}{x^2y})$ is conservative;

$$\left\{\begin{array}{l} \dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial y}(xy^2) - \dfrac{\partial}{\partial x}(x^2y) = 2xy - 2xy = 0 \checkmark \\ \Omega = \mathbb{R}^2 \text{ is simply-connected} \checkmark \end{array}\right.$$

$\Downarrow$

$\vec{F}$ is conservative, i.e. $\exists\, \phi : \Omega \to \mathbb{R}$ s.t. $\vec{F} = \nabla\phi$.

<u>Q</u>: Find ϕ.

$$\phi(x,y) = \int xy^2\,dx + g(y) = \frac{x^2y^2}{2} + g(y)$$

$$\frac{\partial\phi}{\partial y} = x^2y + g'(y) \overset{!}{=} x^2y \implies g'(y) = 0$$

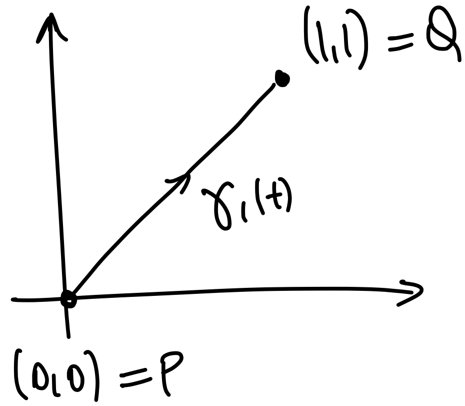
$$\implies g(y) = c.$$

So $\phi(x,y) = \dfrac{x^2y^2}{2} + c$ is a potential for $\vec{F}$, hence

$$\int_{\gamma_1} \vec{F}\,d\gamma_1 = \int_{\gamma_2} \vec{F}\,d\gamma_2 = \int_{\gamma_3} \vec{F}\,d\gamma_3 = \phi(1,1) - \phi(0,0)$$

$$= \left(\frac{1}{2} + c\right) - \left(\frac{0}{2} + c\right) = \boxed{\frac{1}{2}}$$

Verifying that $\int_{(0,0)}^{(1,1)} \vec{F} = \frac{1}{2}$ for a specific path $\gamma_1(t)$.



$$\gamma_1(t) = (1-t)P + tQ$$
$$= (1-t)(0,0) + t(1,1)$$
$$= (t,t), \quad t\in[0,1]$$
$$\gamma_1'(t) = (1,1).$$

$$\int_{\gamma_1} \vec{F}\, d\gamma_1 = \int_0^1 \langle \vec{F}(\gamma_1(t)), \gamma_1'(t)\rangle\, dt = \int_0^1 \langle (t^3, t^3), (1,1)\rangle dt$$
$$= \int_0^1 2t^3\, dt = 2\int_0^1 t^3\, dt = 2\cdot \frac{t^4}{4}\Big|_0^1 = \frac{2}{4} = \boxed{\frac{1}{2}}$$

<u>Exercise</u>: Try doing the same along the path $\gamma_2(t)$ to see that the line integral gives $1/2$.

<u>Theorem</u>: Let $\vec{F}: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. The following are equivalent:

(i) $\exists \phi$ s.t. $\vec{F} = \nabla\phi$, i.e., $\vec{F}$ is conservative

(ii) $\int_\gamma \vec{F}\, d\gamma$ depends only on the endpoints of γ (and not the path γ itself)

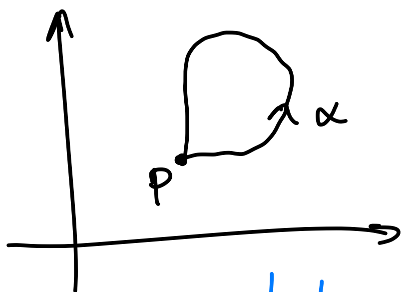
(iii) $\int_\alpha \vec{F}\, d\alpha = 0$ for all closed paths α.

Sketch of pf: (i) $\Rightarrow$ (ii) is the F.T.C. (Line Integrals)

$$\int_\gamma \nabla\phi \, d\gamma = \phi(\gamma(b)) - \phi(\gamma(a)) \quad \checkmark$$

(ii) $\Rightarrow$ (iii) Since $\int_\gamma \vec{F}\, d\gamma$ depends only on the endpoints of γ, and $\int_P \vec{F}\, d\gamma = 0$ and the constant curve $\vec{r}(t) = P$ has the same endpoints as α, it follows that

$$\int_\alpha \vec{F}\, d\alpha = 0.$$



This is the constant curve $\vec{r}(t) = P$, $t \in [0,1]$

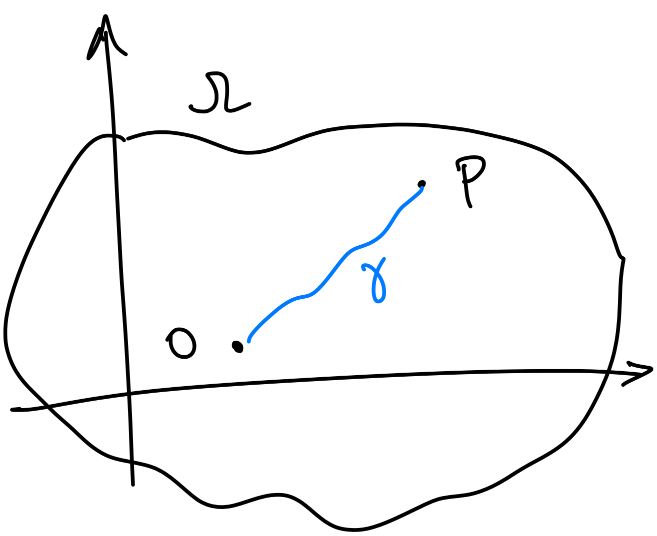
$$\int_P \vec{F}\, dt = \int_0^1 \langle \vec{F}(P), \vec{0} \rangle \, dt = 0.$$

Next: (ii) $\Rightarrow$ (i). To construct a potential $\phi : \Omega \to \mathbb{R}$ for $\vec{F}$, proceed as follows:

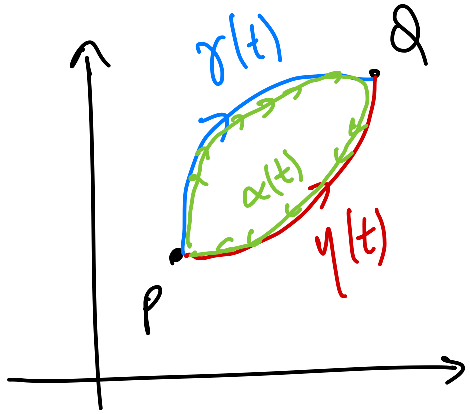
- Choose an "origin" $O \in \Omega$
- Given $P \in \Omega$, define

$$\phi(P) = \int_O^P \vec{F}$$

where we use any curve joining O to P to compute the above line integral. This is well-def. because (ii) holds.



Finally (iii) $\Longrightarrow$ (ii). Need to show that given $P, Q \in \Omega$ and paths $\gamma(t)$, $\eta(t)$ from P to Q,



$$\int_\gamma \vec{F}\, d\gamma = \int_\eta \vec{F}\, d\eta$$

Consider α to be the concatenation of γ and $-\eta$, which is a closed path. By (iii),

$$0 = \int_\alpha \vec{F}\, d\alpha = \int_\gamma \vec{F}\, d\gamma + \int_{-\eta} \vec{F}\, d\eta$$

$$= \int_\gamma \vec{F}\, d\gamma - \int_\eta \vec{F}\, d\eta$$

So $\displaystyle\int_\gamma \vec{F}\, d\gamma = \int_\eta \vec{F}\, d\eta.$ $\square$

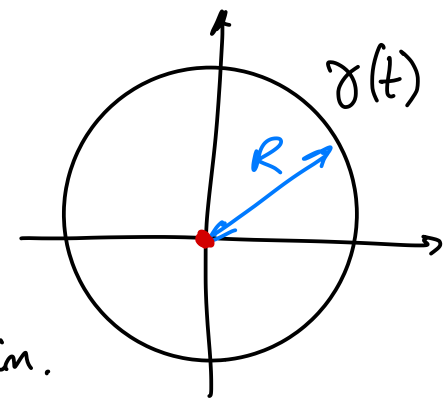
<u>Added after the video:</u> The "only" details I skipped were to show that $\phi(P) = \int_0^P \vec{F}$ is actually a potential for $\vec{F}$, that is, satisfies $\nabla\phi = \vec{F}$, and this is where "most" of the work is in proving the above theorem.

Revisiting the example of $\vec{F}: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ given by

$$\vec{F}(x,y) = \left(\underbrace{-\frac{y}{x^2+y^2}}_{=M}, \underbrace{\frac{x}{x^2+y^2}}_{=N} \right).$$

⚠ $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ is not simply-connected.

From last lecture:



- $\displaystyle\int_\gamma \vec{F}\, d\gamma = 2\pi$ where $\gamma(t)$ is a circle of radius R centered at the origin.

- $\displaystyle\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0.$

By the above Theorem, since $\displaystyle\int_\gamma \vec{F}\, d\gamma \neq 0$ for some closed path γ, (iii) does not hold. Hence (i) also does not hold, that is, $\nexists\, \phi$ such that $\vec{F} = \nabla\phi$.

This shows that $\vec{F}$ is <u>not</u> conservative.

<u>Note</u>: This example shows that the hypothesis of Ω being simply-connected is <u>necessary</u> on the Theorem that states "if Ω is simply-conn. then $\vec{F} = (M, N)$ on Ω is conservative if and only if $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 0$,