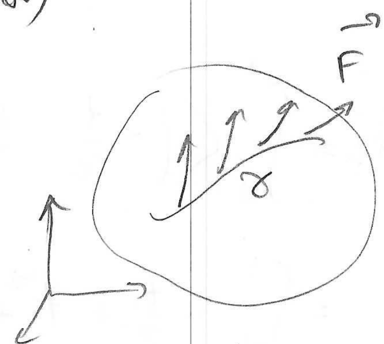
Today: Line Integrals (of vector fields)

$\gamma : [a,b] \to \mathbb{R}^n$ curve

$\vec{F} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ vector field

$(\gamma([a,b]) \subset \Omega)$



$$\int_\gamma \vec{F}\, d\gamma = \int_a^b \langle \vec{F}(\gamma(t)), \gamma'(t) \rangle\, dt$$

Ex: $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$\vec{F}(x,y) = (2x, x+y)$

$$\int_\gamma \vec{F}\, d\gamma = \int_0^{2\pi} \langle (2\cos t, \cos t + \sin t), (-\sin t, \cos t) \rangle\, dt$$

$$= \int_0^{2\pi} -\sin t \cos t + \cos^2 t\ dt$$

$$= \int_0^{2\pi} \frac{1 + \cos 2t}{2}\, dt - \int_0^{2\pi} \frac{\sin 2t}{2} = \frac{2\pi}{2} - \frac{1}{2} \cdot 0 = \pi.$$

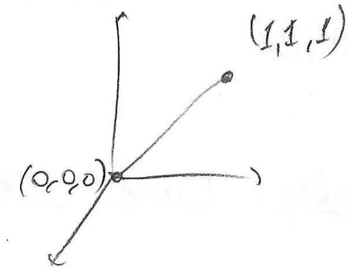
$\vec{G}(x,y) = (1,0) \Rightarrow \int_\gamma \vec{G}\, d\gamma = \int_0^{2\pi} -\sin t = 0$

$\vec{H}(x,y) = (x,y) \Rightarrow \int_\gamma \vec{H}\, d\gamma = \int_0^{2\pi} \underbrace{\langle (\cos t, \sin t), (-\sin t, \cos t) \rangle}_{=0} = 0$

$\vec{I}(x,y) = (-y, x) \Rightarrow \int_\gamma \vec{I}\, d\gamma = \int_0^{2\pi} 1 = 2\pi.$

$$\vec{F}(x,y,z) = \left(-\frac{x}{2},\ xyz,\ z^2+x\right)$$

$$\gamma(t) = (t,t,t), \quad t\in[0,1]$$

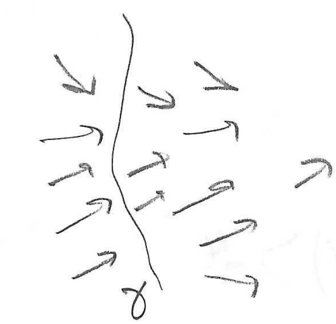


$$\int_\gamma \vec{F}\, d\gamma = \int_0^1 \left\langle \left(-\frac{t}{2},\ t^3,\ t^2+t\right),(1,1,1)\right\rangle dt$$

$$= \int_0^1 -\frac{t}{2} + t^3 + t^2 + t \; dt = \int_0^1 t^3 + t^2 + \frac{t}{2}\; dt$$

$$= \frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{4}\Big|_0^1 = \frac{1}{4}+\frac{1}{3}+\frac{1}{4} = \frac{1}{2}+\frac{1}{3} = \frac{3+2}{6} = \boxed{\frac{5}{6}}$$

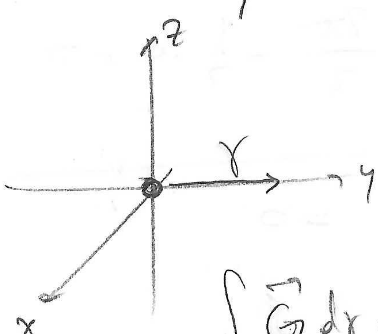
Interpretation: $\vec{F}$ = force $\quad \vec{\gamma}$ = trajectory



$W = \int_\gamma \vec{F}\, d\gamma$ is the work done by an object moving in trajectory γ under the force field $\vec{F}$

Ex: Gravity around the earth:

$$\vec{G}(x,y,z) = -M\,\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$



$$\vec{\gamma}(t) = (0,t,0), \quad t\in[1,2], \quad m=1$$

$$\int_\gamma \vec{G}\, d\gamma = \int_1^2 \left\langle \left(0, -\frac{t}{t^3}, 0\right),(0,1,0)\right\rangle dt = \int_1^2 -\frac{1}{t^2}\, dt$$

$$= \frac{1}{t}\Big|_1^2 = \frac{1}{2} - 1 = -\frac{1}{2}$$ ← negative b/c in opposite direction of displacement!

$$= \int_0^1 3t^5 - 5t^4 - 2t^3 + 3t^2 + t \; dt = \frac{1}{2}.$$

Lecture 23 started here:

Discuss HW prob. v.

$$F(x,y) = \Big(\underbrace{-\frac{y}{x^2+y^2}}_{M}, \underbrace{\frac{x}{x^2+y^2}}_{N} \Big) \qquad \Omega = \mathbb{R}^2 \setminus \{(0,0)\}$$

not simply connected.

$$\frac{\partial M}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}$$

but: not conservative!

$$\int_{S^1} \vec{F}\, d\gamma \neq 0 \qquad \int_{S^1} \vec{F} = 2\pi \neq 0.$$