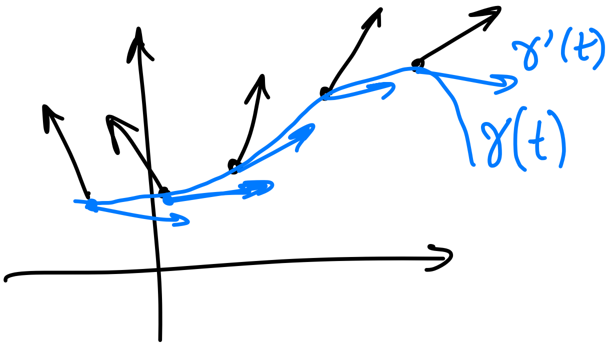
Line integrals of vector fields

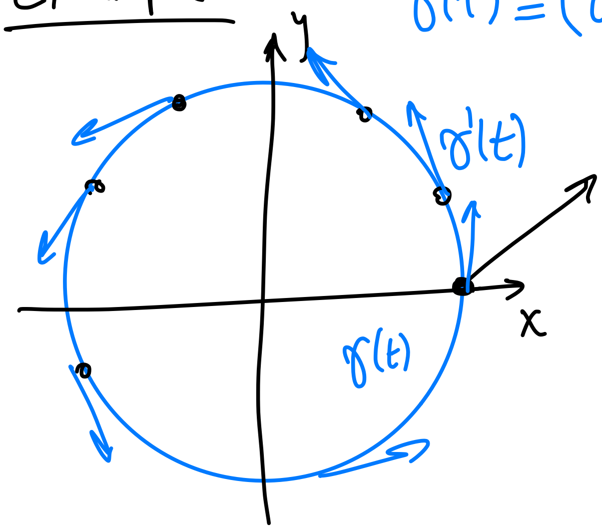
Recall: a vector field is a function $\vec{F}: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$



Def: Given $\gamma: [a,b] \to \mathbb{R}^n$ a curve, one can define the line integral of $\vec{F}$ along γ:

$$\int_\gamma \vec{F}\, d\gamma = \int_a^b \langle \vec{F}(\gamma(t)), \gamma'(t) \rangle \, dt$$

Example: $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$



$\gamma'(t) = (-\sin t, \cos t)$.

$\vec{F}(x,y) = (2x, x+y)$

$\vec{F}(\gamma(t)) = (2\cos t, \cos t + \sin t)$

$$\int_\gamma \vec{F}\, d\gamma = \int_0^{2\pi} \langle \vec{F}(\gamma(t)), \gamma'(t) \rangle \, dt$$

$$= \int_0^{2\pi} \langle (2\cos t, \cos t + \sin t), (-\sin t, \cos t) \rangle \, dt$$

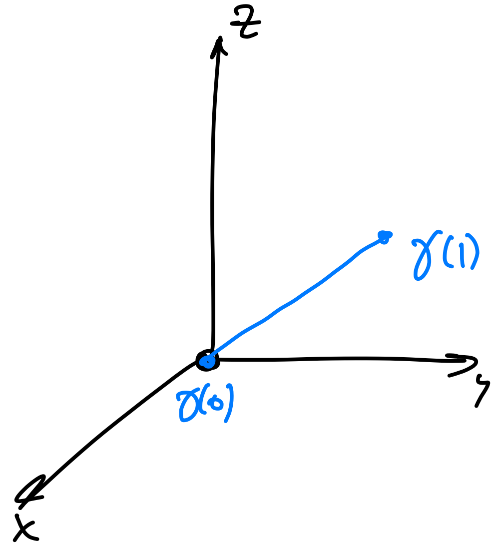
$$= \int_0^{2\pi} -2\cos t \sin t + \cos^2 t + \cos t \sin t \, dt = \int_0^{2\pi} \cos^2 t - \cos t \sin t \, dt$$

$$= \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt - \int_0^{2\pi} \frac{\sin 2t}{2}\,dt = \boxed{\pi.}$$

Another example, in 3D:

$$\vec{F}(x,y,z) = \left(-\frac{x}{2},\ xyz,\ z^2+x\right)$$

$$\gamma(t) = (t,t,t), \quad t \in [0,1]$$



A: $\vec{F}(\gamma(t)) = \left(-\frac{t}{2},\ t^3,\ t^2+t\right)$

$$\gamma'(t) = (1,1,1)$$

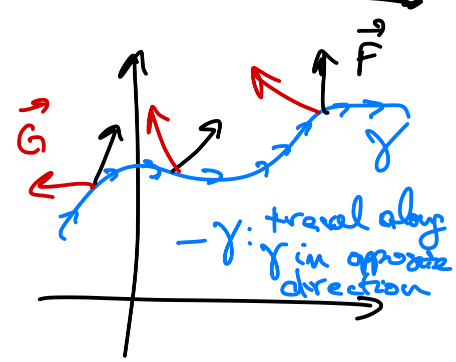
$$\int_\gamma \vec{F}\,d\gamma = \int_0^1 \langle \vec{F}(\gamma(t)), \gamma'(t)\rangle\,dt = \int_0^1 \left\langle \left(-\frac{t}{2}, t^3, t^2+t\right), (1,1,1)\right\rangle dt$$

$$= \int_0^1 -\frac{t}{2} + t^3 + t^2 + t\ dt = \int_0^1 t^3 + t^2 + \frac{t}{2}\,dt$$

$$= \left(\frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{4}\right)\Bigg|_0^1 = \frac{1}{4} + \frac{1}{3} + \frac{1}{4} = \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

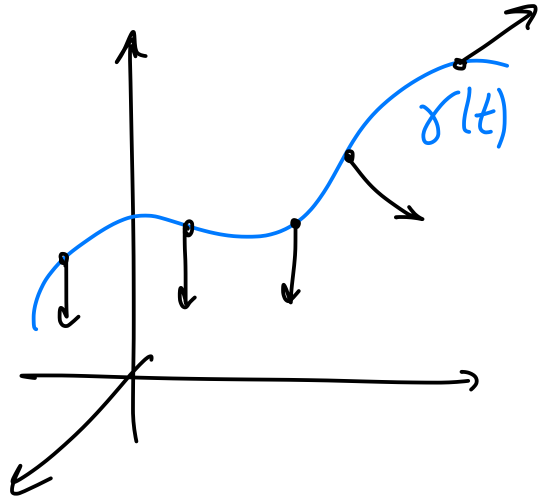
Basic Properties of Line Integrals:

1) $\int_\gamma a\vec{F} + b\vec{G}\ d\gamma = a\int_\gamma \vec{F}\,d\gamma + b\int_\gamma \vec{G}\,d\gamma$



2) $\int_{-\gamma} \vec{F}\,d\gamma = -\int_\gamma \vec{F}\,d\gamma$

Physical Interpretation:



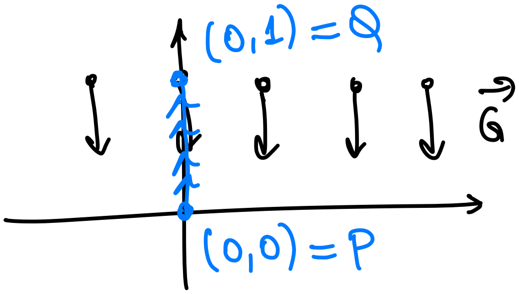
$\vec{F}$ = Force

γ = trajectory of an object

$$W = \int_\gamma \vec{F}\, d\gamma \qquad \text{work}$$

Example: $\vec{G}(x,y) = (0,-1)$, $\gamma(t) = (1-t)P + tQ = (0,t)$

$t \in [0,1]$



$$\int_\gamma \vec{G}\, d\gamma = \int_0^1 \langle (0,-1), (0,1) \rangle \, dt$$

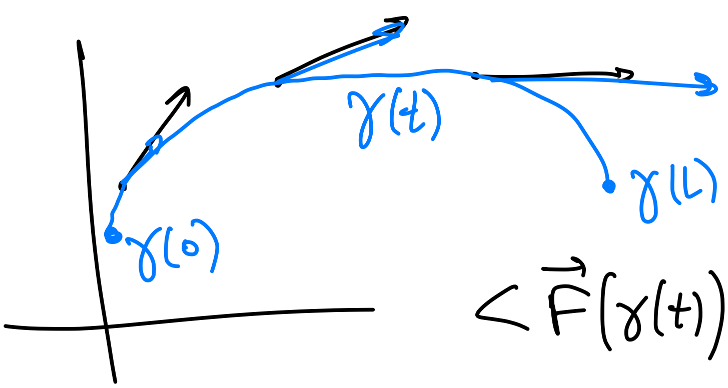
$$= \int_0^1 -1 \, dt = -1.$$

Negative

Need to exert this amount of work to move along trajectory.

Note: $\displaystyle\int_{-\gamma} \vec{G}\, d\gamma = -\int_\gamma \vec{G}\, d\gamma = 1$

Compare with: $W = \|\vec{F}\| \cdot d$ ← Is what we get for the work if force has constant magnitude $\|\vec{F}\|$ and movement is precisely in the direction of the force.



$$\langle \vec{F}(\gamma(t)), \gamma'(t) \rangle = \|\vec{F}(\gamma(t))\| \cdot \|\gamma'(t)\| \cdot \underbrace{\cos\theta}_{1}$$

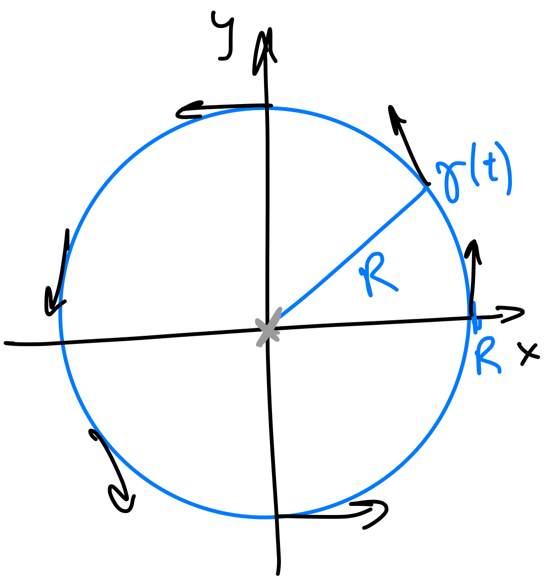
$$= \|\vec{F}\| \cdot \|\gamma'(t)\|$$

$$W = \int_\gamma \vec{F}\, d\gamma = \int_0^L \langle \vec{F}(\gamma(t)), \gamma'(t) \rangle\, dt = \int_0^L \|\vec{F}\| \cdot \|\gamma'(t)\|\, dt$$

$$= \|\vec{F}\| \cdot \int_0^L \|\gamma'(t)\|\, dt = \|\vec{F}\| \cdot d$$

arc length of the curve $\gamma(t)$ between $t=0$ and $t=L$.

An important example: $\vec{F}: \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}^2$



$$\vec{F}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Note. $\langle (x,y), \vec{F}(x,y) \rangle = 0$

$$\|\vec{F}(x,y)\|^2 = \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$$

so $\|\vec{F}(x,y)\| = \dfrac{1}{\|(x,y)\|}$

$\gamma(t) = (R\cos t, R\sin t)$, $t \in [0, 2\pi]$

$\gamma'(t) = (-R\sin t, R\cos t)$

$$\vec{F}(\gamma(t)) = \left(-\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2} \right) = \left(-\frac{\sin t}{R}, \frac{\cos t}{R} \right)$$

$$\int_\gamma \vec{F}\, d\gamma = \int_0^{2\pi} \langle \vec{F}(\gamma(t)), \gamma'(t) \rangle\, dt =$$

$$= \int_0^{2\pi} \left\langle \left(-\frac{\sin t}{R}, \frac{\cos t}{R}\right), (-R\sin t, R\cos t) \right\rangle dt$$

$$= \int_0^{2\pi} \underbrace{\sin^2 t + \cos^2 t}_{1} \, dt = \boxed{2\pi}$$

<u>Q:</u> Compute $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ where $\vec{F} = (M, N)$

$$M = -\frac{y}{x^2+y^2} \qquad N = \frac{x}{x^2+y^2}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1.(x^2+y^2) - x.2x}{(x^2+y^2)^2} = \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = -\frac{1.(x^2+y^2) - y.2y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} = \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$$

So $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$

Since $\mathbb{R}^2 \setminus \{(0,0)\}$ is <u>not</u> simply-connected, even though $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, we <u>cannot</u> conclude that $\vec{F} = (M,N)$ is conservative. As it turns out, the above $\vec{F}$ is <u>not</u> conservative.