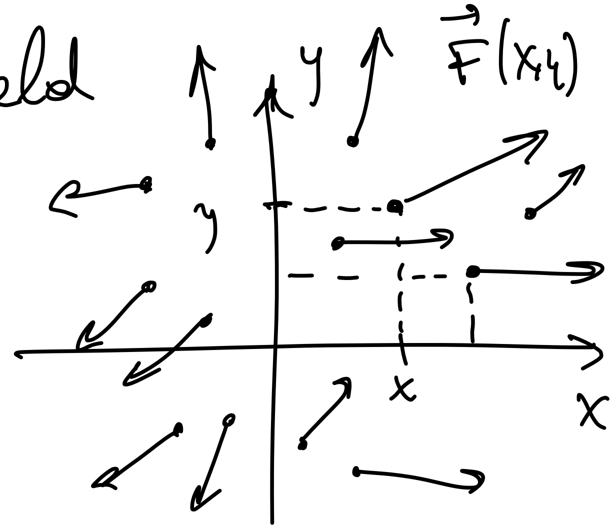
MAT226 Lecture 23 4/27/2020

Vector fields:

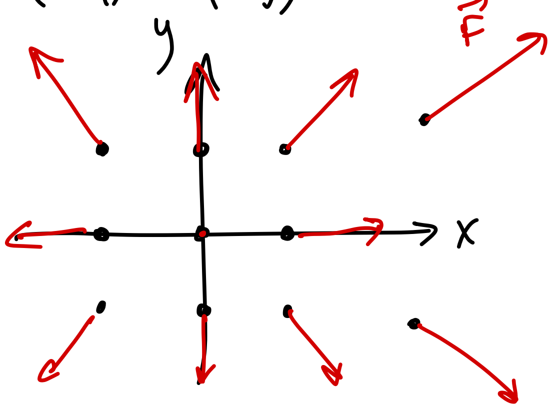
$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field

$$\underbrace{(x,y)}_{\text{point}} \longmapsto \underbrace{\vec{F}(x,y)}_{\text{vector}}$$

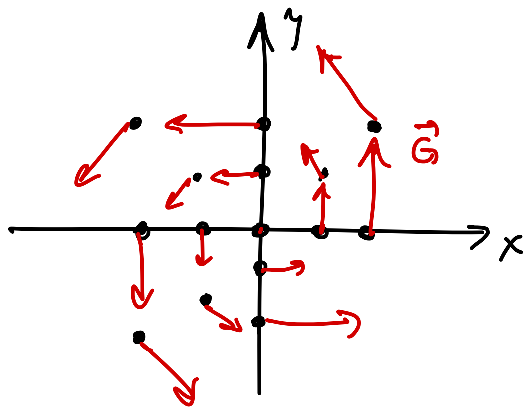


Examples:

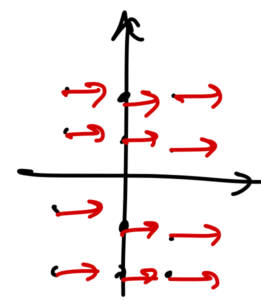
$\vec{F}(x,y) = (x,y)$



$\vec{G}(x,y) = (-y, x)$



$\vec{H}(x,y) = (1,0)$



Important class of vector fields: $\vec{F} = \nabla f$

Those that can be written as the gradient of some $f(x,y)$.

Def: A vector field $\vec{F}$ is called <u>conservative</u> if there exists a scalar-valued function f such that $\vec{F} = \nabla f$. Such a function f is called a <u>potential</u> for $\vec{F}$.

E.g.: $f(x,y) = 3x^2 + 4xy - y^3 + 1.$

$$\nabla f(x,y) = \left(6x + 4y,\ 4x - 3y^2\right)$$

So $\vec{F}(x,y) = \left(\underbrace{6x+4y}_{M},\ \underbrace{4x-3y^2}_{N}\right)$ is conservative!

Notice: Writing $\vec{F} = M\hat{\imath} + N\hat{\jmath} = (M,N)$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

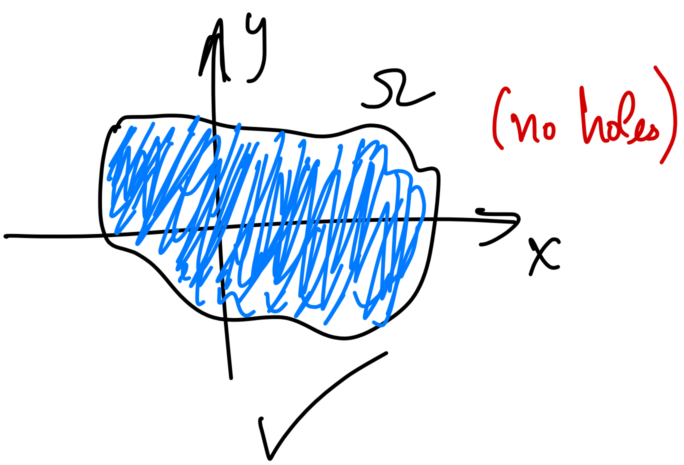
$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$$

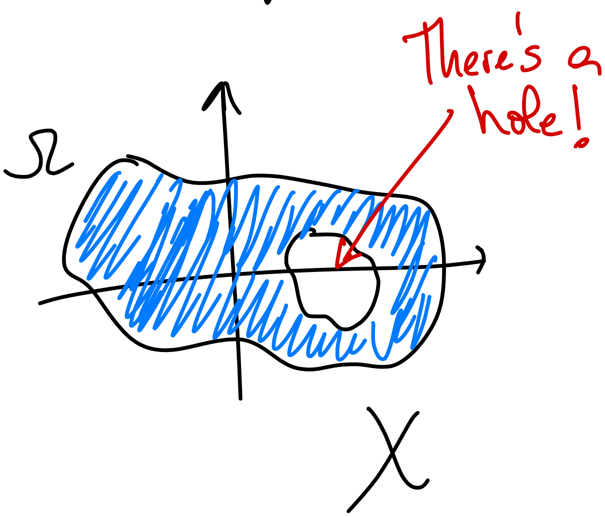
So: the condition that $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ is necessary for $\vec{F} = (M,N)$ to be conservative.

Q: Is it also sufficient?

A: Yes, provided that the domain of $\vec{F}$ is simply-connected.

Def: $\Omega \subset \mathbb{R}^2$ is <u>simply-connected</u> if "it has no holes". In other words, any closed curve in Ω can be continuously deformed to a point.





Using the above notation;

<u>Thm.</u> Let $\Omega \subset \mathbb{R}^2$ be a <u>simply-connected</u> domain. A vector field $\vec{F}: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{F} = (M, N)$ is conservative <u>if and only if</u>

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0.$$

Ex: Is $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ conservative?

The domain of all these vect. fields is $\Omega = \mathbb{R}^2$ which is simply-connected

a) $\vec{F}(x,y) = (\underbrace{x^2 y}_{M}, \underbrace{xy+1}_{N})$

$$\frac{\partial M}{\partial y} = x^2 \neq \frac{\partial N}{\partial x} = y$$ NOT CONSERVATIVE!

b) $\vec{F}(x,y) = (\underbrace{2xy}_{M}, \underbrace{x^2 - y}_{N})$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} = 2x$$ IS CONSERVATIVE!

c) $\vec{F}(x,y) = (2x, y)$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} = 0$$ IS CONSERVATIVE!

Q: How to find the potential of a conservative vec. field?

$$\vec{F}(x,y) = (\underbrace{M(x,y)}_{= \frac{\partial f}{\partial x}(x,y)}, \underbrace{N(x,y)}_{= \frac{\partial f}{\partial y}(x,y)}) = \nabla f(x,y)$$

A:

1. $f(x,y) = \int M(x,y)\,dx + g(y)$

2. Differentiate both sides in y, and compare with $N = \frac{\partial f}{\partial y}$ to find $g(y)$.

Note: Can also reverse roles of x and y:

1'. $f(x,y) = \int N(x,y)\, dy + h(x)$

2'. Differentiate both sides in x, and compare with $M = \frac{\partial f}{\partial x}$ to find $h(x)$.

Ex: $\vec{F}(x,y) = (2xy, x^2 - y)$

1'. $f(x,y) = \int (x^2 - y)\, dy + h(x) = x^2 y - \frac{y^2}{2} + h(x)$

2'. $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 y - \frac{y^2}{2} + h(x)\right) = 2xy + h'(x)$

$M = 2xy \quad \Rightarrow h'(x) = 0.$

$\Rightarrow h(x) = c$

$$\boxed{\text{Ans: } f(x,y) = x^2 y - \tfrac{1}{2}y^2 + c}$$

Note: If $f(x,y)$ is a potential for $\vec{F}(x,y)$, so is $f(x,y) + \text{const}$ for any constants!

We could also find this potential by using the other order, as follows:

1. $f(x,y) = \int 2xy\, dx + g(y) = x^2 y + g(y)$

2. $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y + g(y)\right) = x^2 + g'(y)$

$N = x^2 - y \implies g'(y) = -y$

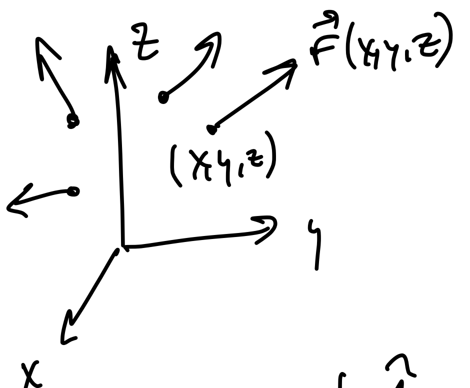
$\implies g(y) = -\frac{y^2}{2} + c$

$$\boxed{f(x,y) = x^2 y - \frac{y^2}{2} + c}$$

Potential for $\vec{F}(x,y) = (2x, y)$: $f(x,y) = x^2 + \frac{y^2}{2} + c$

<u>How about 3D vector fields?</u>

$\vec{F} : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$



<u>Q</u>: When is $\vec{F} = (M, N, P)$ conservative?

<u>Def</u>: The <u>curl</u> of $\vec{F} = (M, N, P)$ is the vector field given by

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

A:

Thm. Let $\Omega \subset \mathbb{R}^3$ be a simply-connected domain.
A vector field $\vec{F} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is conservative if and only if $\operatorname{curl} \vec{F} = \nabla \times \vec{F} = 0$.

Ex: $\vec{F}(x,y,z) = (\underbrace{2xy}_{M}, \underbrace{x^2+z^2}_{N}, \underbrace{2yz}_{P})$ $\qquad \Omega = \mathbb{R}^3$ simply-conn. ✓

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+z^2 & 2yz \end{vmatrix} = (2z - 2z, 0, 2x - 2x)$$
$$= (0,0,0).$$

$\Longrightarrow \vec{F}$ is conservative!

Q: Find a potential $f(x,y,z)$ s.t. $\vec{F} = \nabla f$.

$$f(x,y,z) = \int 2xy \, dx + g(y,z) = x^2 y + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 y + g(y,z)\right) = x^2 + \frac{\partial g}{\partial y} \overset{!}{=} x^2 + z^2 = N$$

$$\Longrightarrow \frac{\partial g}{\partial y} = z^2 \Longrightarrow g(y,z) = yz^2 + h(z)$$

So now we know

$$f(x,y,z) = x^2y + yz^2 + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2y + yz^2 + h(z)\right) = 2yz + h'(z) \overset{!}{=} 2yz = P$$

$$\Longrightarrow h'(z) = 0 \Longrightarrow h(z) = c$$

We conclude the potential is:

$$\boxed{f(x,y,z) = x^2y + yz^2 + c}$$

Check: Compute ∇f and note it matches $\vec{F}(x,y,z)$!