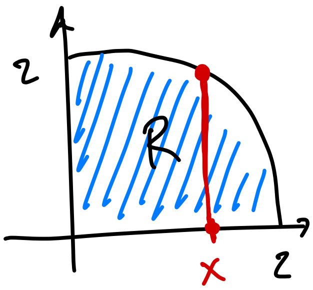
An example:



$$R: \quad x^2+y^2 \le 4, \quad x \ge 0, \quad y \ge 0.$$

$$0 \le x \le 2 \qquad 0 \le y \le \sqrt{4-x^2}$$

$$\iint_R x+y \, dA = \int_0^2 \int_0^{\sqrt{4-x^2}} (x+y)\, dy\, dx$$

$$= \int_0^2 \left(xy + \frac{y^2}{2} \right) \Bigg|_0^{\sqrt{4-x^2}} dx =$$

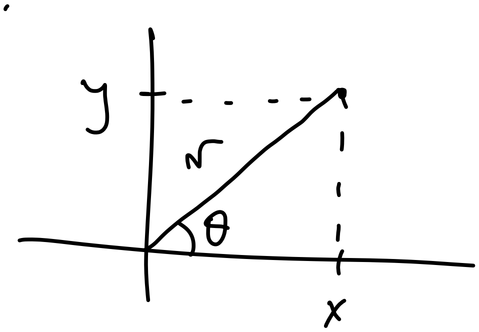
$$= \int_0^2 x\sqrt{4-x^2} + \frac{4-x^2}{2} \, dx$$

$$= \left(-\frac{(4-x^2)^{3/2}}{3} + 2x - \frac{x^3}{6} \right) \Bigg|_0^2$$

$$= \left(-0 + 4 - \frac{8}{6} \right) - \left(-\frac{8}{3} \right) = 4 - \frac{4}{3} + \frac{8}{3} = 4 + \frac{4}{3} = \boxed{\frac{16}{3}}$$

Replace x, y with polar coordinates r, θ:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



Using product rule:

$$\begin{cases} dx = \cos\theta\, dr + r(-\sin\theta)\, d\theta \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta \end{cases}$$

Don't worry about details of this computation

$$\boxed{dA} = dx\,dy = r\cos^2\theta\, dr\, d\theta - r\sin^2\theta\, d\theta\, dr$$

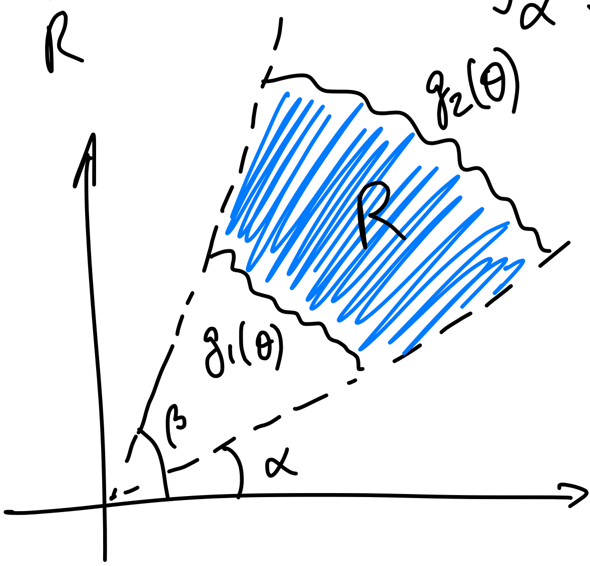
$$= r\cos^2\theta\, dr\, d\theta + r\sin^2\theta\, dr\, d\theta$$

$$= r\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}\, dr\, d\theta = \boxed{r\, dr\, d\theta}$$

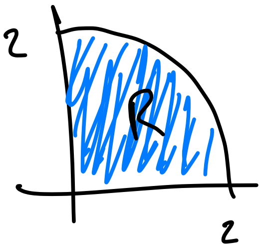
dA in polar coordinates will always be this!

Integrals in Polar Coordinates:

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



Revisiting the example from the beginning:



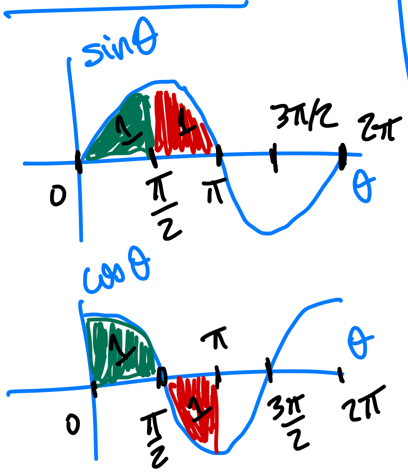
$$\iint_R x+y \, dA = \int_0^{\pi/2}\int_0^2 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left(\int_0^2 r^2(\cos\theta+\sin\theta)\,dr\right)d\theta$$

$$= \int_0^{\pi/2}(\cos\theta+\sin\theta)\left.\frac{r^3}{3}\right|_0^2 d\theta = \frac{8}{3}\int_0^{\pi/2}\cos\theta+\sin\theta\, d\theta$$

$$= \frac{8}{3}\Big(\underbrace{\int_0^{\pi/2}\cos\theta\, d\theta}_{1}\Big) + \frac{8}{3}\Big(\underbrace{\int_0^{\pi/2}\sin\theta\, d\theta}_{1}\Big) = \boxed{\frac{16}{3}}$$

Remember:



Each "half lump" has area $=1$!

In particular:

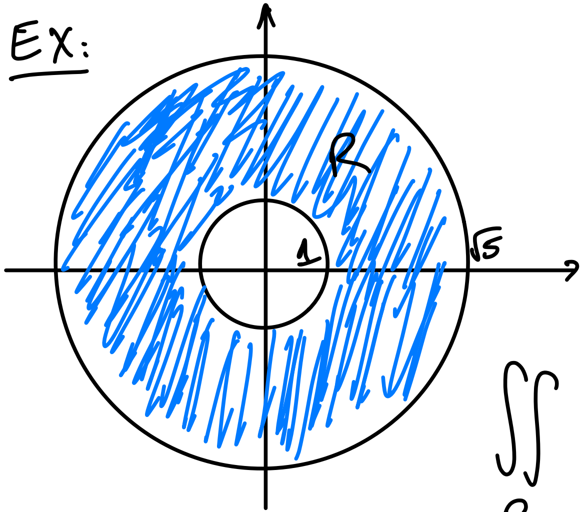
$$\int_0^{\pi/2}\sin\theta\, d\theta = \int_0^{\pi/2}\cos\theta\, d\theta = 1$$

$$\int_0^{\pi}\sin\theta\, d\theta = 2, \qquad \int_0^{\pi}\cos\theta\, d\theta = 0$$

$$\int_0^{\frac{3\pi}{2}}\sin\theta\, d\theta = 1, \qquad \int_0^{\frac{3\pi}{2}}\cos\theta\, d\theta = -1$$

$$\int_0^{2\pi}\sin\theta\, d\theta = \int_0^{2\pi}\cos\theta\, d\theta = 0.$$

Ex:



$R: \quad 1 \le x^2+y^2 \le 5$

In polar coord:

$0 \le \theta \le 2\pi \qquad 1 \le r \le \sqrt{5}$

$$\iint_R x^2+y \, dA = \int_0^{2\pi}\int_1^{\sqrt{5}} \left(r^2\cos^2\theta + r\sin\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{5}} r^3\cos^2\theta + r^2\sin\theta \, dr\,d\theta = \int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right)\Bigg|_1^{\sqrt{5}} d\theta$$

$$= \int_0^{2\pi} \frac{25}{4}\cos^2\theta + \frac{5\sqrt{5}}{3}\sin\theta - \frac{1}{4}\cos^2\theta - \frac{1}{3}\sin\theta \quad d\theta$$

$$= 6\int_0^{2\pi}\cos^2\theta\,d\theta + \frac{5\sqrt{5}-1}{3}\underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{=0}$$

$$= 6\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta = \frac{6}{2}\int_0^{2\pi} 1\,d\theta = 3.2\pi = \boxed{6\pi}$$

Remember:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Use these to integrate!

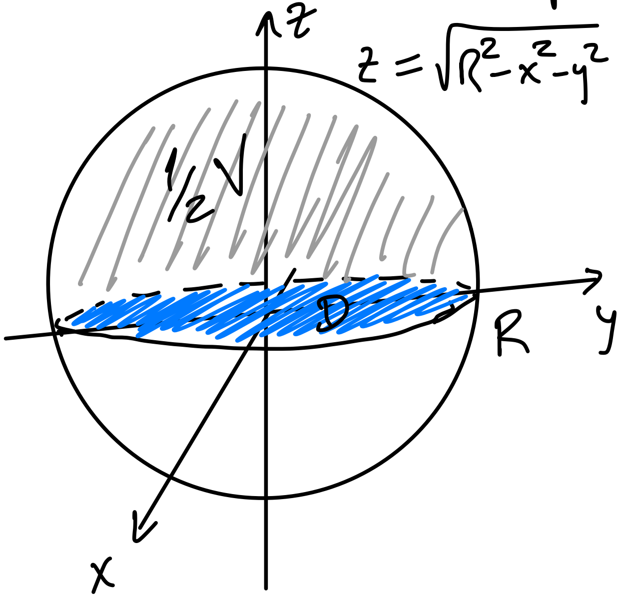
$$\cos^2\theta + \sin^2\theta = 1$$

$$-\cos^2\theta \pm \sin^2\theta = -\cos 2\theta$$

Add the above: $2\cos^2\theta = 1 + \cos 2\theta$

Subtract the above: $2\sin^2\theta = 1 - \cos 2\theta$

Ex: Volume of a sphere of radius $R = \frac{4}{3}\pi R^3$



$x^2 + y^2 + z^2 = R^2$

D: $x^2 + y^2 \le R^2$

In polar coord: $0 \le \theta \le 2\pi$
$0 \le r \le R$

$$\frac{1}{2}V = \iint_D \sqrt{R^2 - x^2 - y^2}\, dA$$

$$= \int_0^{2\pi}\int_0^R \sqrt{R^2 \underbrace{- r^2\cos^2\theta - r^2\sin^2\theta}_{= -r^2}}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^R \sqrt{R^2 - r^2}\; r\,dr\,d\theta = 2\pi\int_0^R r\sqrt{R^2 - r^2}\,dr$$

$u = R^2 - r^2$
$du = -2r\,dr$

$$= -2\pi \left.\frac{(R^2 - r^2)^{3/2}}{3}\right|_0^R = -\frac{2\pi}{3}\left[0 - R^3\right] = \frac{2\pi R^3}{3}$$

$$\Rightarrow V = \frac{4\pi R^3}{3}$$