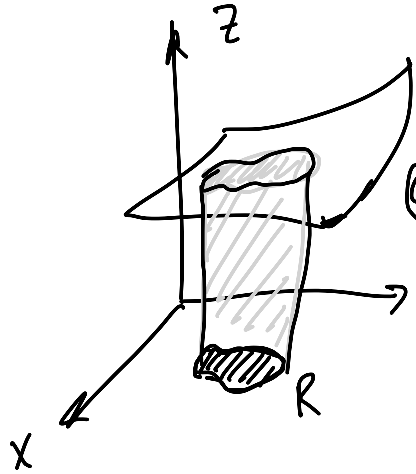
Double integrals to find volumes:

From last time:

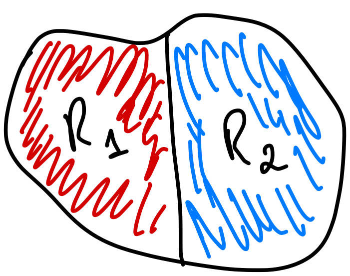
$z = f(x,y)$



$$\mathrm{Vol}\begin{pmatrix}\text{Region under} \\ \text{graph of } f(x,y) \\ \text{over } R\end{pmatrix} = \iint_R f(x,y)\, dA$$

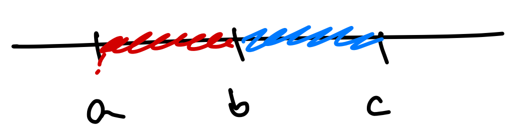
Properties: (1) $\iint_R af + bg \, dA = a\iint_R f\,dA + b\iint_R g\,dA$ **Linearity** $(a,b \in \mathbb{R})$

(2) $\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA$



$R = R_1 \cup R_2$

$$\left(\text{cf}: \int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx \right)$$



Ex: Find the volume under the paraboloid

$$z = 4 - x^2 - 2y^2 \quad \text{and above} \quad xy\text{-plane:}$$

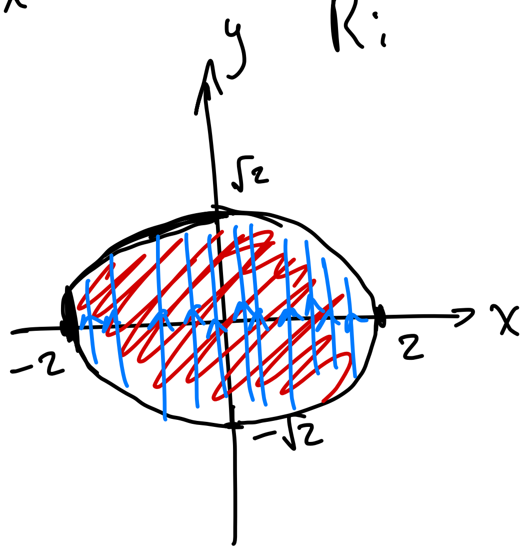
$$\text{Vol} = \iint_R 4 - x^2 - 2y^2 \, dA$$

Parametrize the region R:

$$-2 \le x \le 2$$

$$-\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$$

R:



$$R = \{(x,y) \in \mathbb{R}^2 ; \; x^2 + 2y^2 \le 4\}$$

$$\Updownarrow$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 \le 1$$

Solve for y: $2y^2 = 4 - x^2 \Rightarrow y^2 = \dfrac{4-x^2}{2}$

$$\Rightarrow y = \pm\sqrt{\frac{4-x^2}{2}}$$

$$\iint_R 4 - x^2 - 2y^2 \, dA = \int_{-2}^{2} \left(\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} 4 - x^2 - 2y^2 \, dy \right) dx$$

$$= \int_{-2}^{2} \underbrace{\left(4y - x^2 y - \frac{2y^3}{3}\right) \Bigg|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}} \, dx$$

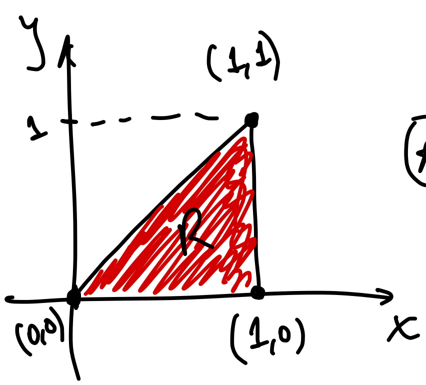
$$= 2\int_{-2}^{2} 4\sqrt{\frac{4-x^2}{2}} - x^2\sqrt{\frac{4-x^2}{2}} - \frac{2}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\,dx$$

$$= 2\int_{-2}^{2} (4-x^2)\left(\frac{4-x^2}{2}\right)^{1/2} - \frac{2}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\,dx$$

$$= \frac{4}{3\sqrt{2}}\int_{-2}^{2}(4-x^2)^{3/2}\,dx = \frac{4}{3\sqrt{2}}\int_{-\pi/2}^{\pi/2} 16\cos^4\theta\,d\theta = \boxed{4\sqrt{2}\pi}$$

$x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$

__Example:__ Find the __volume__ under $z = e^{-x^2}$ over the region R given by the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$.



__Parametrize R:__

(A)

#s → $0 \le x \le 1$

fcts of x → $0 \le y \le x$

$dA = dy\,dx$

(B)

#s → $0 \le y \le 1$

fcts of y → $y \le x \le 1$

$dA = dx\,dy$

(A) $$\text{Vol} = \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right) dx$$

$$= \int_0^1 e^{-x^2} \cdot y \Big|_0^x \, dx = \int_0^1 e^{-x^2}(x-0)\, dx$$

$$= \int_0^1 x e^{-x^2}\, dx = -\frac{e^{-x^2}}{2} \Big|_0^1 = \boxed{\frac{e-1}{2e}}$$

$\left\{ \begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array} \right.$

(B) $$\text{Vol} = \iint_R e^{-x^2}\, dA = \int_0^1 \left(\int_y^1 e^{-x^2}\, dx \right) dy$$

Note: e^{-x^2} cannot be integrated (w.r.t. x) by elementary methods.

Upshot: Must use parametrization (A).

Other functions that cannot be integrated by elementary methods: e^{-x^2}, $\frac{\sin x}{x}$, $\frac{1}{\ln x}$