MAT 226 Lecture 15 10/30/2019

Multiple Integrals (Chap 14)

$$\int_1^x \left(2x^2y^{-2}+2y\right)dy = \left(-\frac{2x^2}{y}+y^2\right)\Big|_1^x$$

$$= \left(-\frac{2x^2}{x}+x^2\right)-\left(-2x^2+1\right)$$

$$= 3x^2-2x-1.$$

$$\int_1^2\int_1^x \left(2x^2y^{-2}+2y\right)dy\,dx = \int_1^2 3x^2-2x-1\,dx$$

$$= \left(x^3-x^2-x\right)\Big|_1^2$$

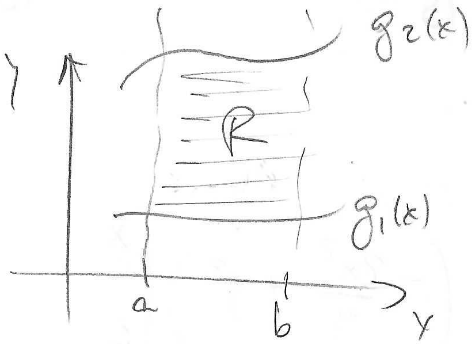
$$= (8-4-2)-(-1) = \boxed{3.}$$

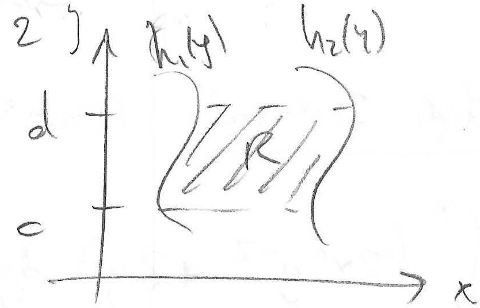
Iterated integrals:

1) $$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)dx$$

2) $$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$$

Note: 1)



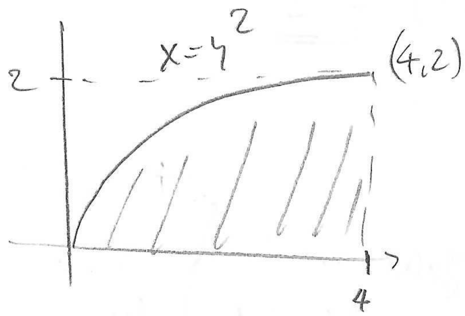


R = Region of integration

$$\iint_R f(x,y)\, dA \qquad dA = \underset{①}{dx\,dy} \quad \text{or} \quad dA = \underset{②}{dy\,dx}$$

Note: (Area of region R) $= \iint_R 1.dA$

Example:



$$R = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} 0 \le x \le 4 \\ 0 \le y \le \sqrt{x} \end{array} \right\}$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} 0 \le y \le 2 \\ y^2 \le x \le 4 \end{array} \right\}$$

$$\text{Area} = \iint_R dA = \int_0^4 \int_0^{\sqrt{x}} 1.\, dy\, dx = \int_0^2 \int_{y^2}^4 dx\, dy$$

$$\int_0^4 \sqrt{x}\, dx = \frac{x^{3/2}}{3/2}\Big|_0^4 = \frac{16}{3}$$

$$\int_0^2 4 - y^2\, dy$$

$$\left(4y - \frac{y^3}{3}\right)\Big|_0^2 = \frac{16}{3}$$

How about integrate non constant $f(x,y)$:

$$\iint_R x+y \, dA = \int_0^4 \int_0^{\sqrt{x}} x+y \, dy\, dx$$

$$= \int_0^4 xy + \frac{y^2}{2}\Big|_0^{\sqrt{x}} dx$$

$$= \int_0^4 x\sqrt{x} + \frac{x}{2} \, dx$$

$$= \left(\frac{2}{5} x^{5/2} + \frac{x^2}{4}\right)\Big|_0^4$$

$$= \frac{2}{5}\cdot 2^5 + 4 = \frac{64}{5} + \frac{20}{5} = \boxed{\frac{84}{5}}$$

or

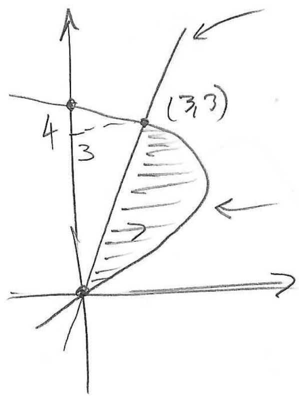
$$\iint_R x+y \, dA = \int_0^2 \int_{y^2}^4 x+y \, dx\, dy$$

$$= \int_0^2 \left(\frac{x^2}{2} + xy\right)\Big|_{y^2}^4 dy = \int_0^2 (8+4y) - \left(\frac{y^4}{2} + y^3\right) dy$$

$$= \int_0^2 8 + 4y - \frac{y^4}{2} - y^3 \, dy = \left(8y + 2y^2 - \frac{y^5}{10} - \frac{y^4}{4}\right)\Big|_0^2$$

$$= 16 + 8 - \frac{32}{10} - 4 = 20 - \frac{32}{10} = \frac{100-16}{5} = \boxed{\frac{84}{5}}$$

⚠ Sometimes one of these options is much better (computationally) than the other!



$x = y$

$x = y(4-y)$

Intersection pt.:

$y = y(4-y) = 4y - y^2$

$y^2 - 3y = 0$

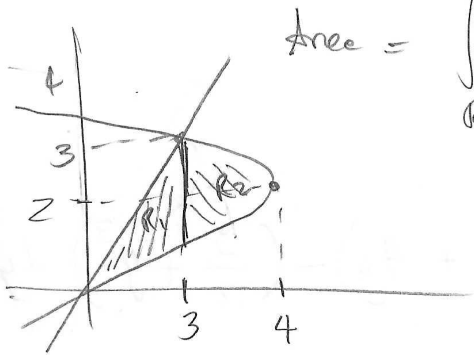
$y(y-3) = 0 \Rightarrow y = 0$ or $y = 3$

$$\text{Area} = \iint_R dA = \int_0^4 \int_y^{y(4-y)} 1 \; dx \cdot dy = \int_0^4 (y(4-y) - y) \; dy$$

$$= \int_0^4 3y - y^2 \; dy = \left(\frac{3y^2}{2} - \frac{y^3}{3}\right)\Big|_0^4 = 3 \cdot 8 - \frac{64}{3}$$

$$= 24 - \frac{64}{3} = \boxed{\frac{8}{3}}$$

Or:



$$\text{Area} = \iint_{R_1} dA + \iint_{R_2} dA =$$

$$x = y(4-y) = 4y - y^2 \Rightarrow y^2 - 4y + x = 0$$

$$\Rightarrow y = \frac{4 \pm \sqrt{16 - 4x}}{2} = 2 \pm \sqrt{4-x}$$

$R_1 = \{(x,y) : 0 \le x \le 3 \;\; \underline{2 - \sqrt{4-x}} \le y \le \underline{x}\}$

$R_2 = \{(x,y) : 3 \le x \le \underline{4}, 2 - \sqrt{4-x} \le y \le 2 + \sqrt{4-x}\}$

$$\iint_{R_1} dA = \int_0^3 \int_{2-\sqrt{4-x}}^{x} dy\,dx = \int_0^3 x - 2 + \sqrt{4-x} \; dx = \frac{19}{6}$$

$$\iint_{R_2} dA = \int_3^4 \int_{2-\sqrt{4-x}}^{2+\sqrt{4-x}} dy\,dx = \int_3^4 2\sqrt{4-x} \; dx = \frac{4}{3}$$

Area $= \frac{9}{2}$